Estimations of Similarity in Formal Concept Analysis of Data with Graded Attributes*

Radim Bělohlávek and Vilém Vychodil

Dept. Computer Science, Palacký University Tomkova 40, CZ-779 00, Olomouc, Czech Republic, {radim.belohlavek, vilem.vychodil}@upol.cz

Summary. We study similarity in formal concept analysis of data tables with graded attributes. We focus on similarity related to formal concepts and concept lattices, i.e. the outputs of formal concept analysis. We present several formulas for estimation of similarity of outputs in terms of similarity of inputs. The results answer some problems which arose in previous investigation as well as some natural questions concerning similarity in conceptual data analysis. The derived formulas enable us to compute an estimation of similarity of concept lattices much faster than one can compute their exact similarity. We omit proofs due to lack of space.

Key words: formal concept analysis, fuzzy logic, similarity, concept lattice, hedge

1 Introduction and problem setting

Formal concept analysis (FCA) is a method for analysis of tabular data describing objects and their attributes [8, 9]. There are two basic outputs of FCA, namely, a concept lattice and attribute implications. A concept lattice is a set of all clusters (called formal concepts) extracted from data, hierarchically ordered by subconcept-superconcept relation. Attribute implications are particular expressions describing certain attribute dependencies. Efficient algorithms are known to compute a concept lattice and a non-redundant set of attribute implications which entail all attribute implications true in data.

In the basic setting, attributes are assumed to be bivalent, i.e. either a given attribute y applies to a given object x (indicated by 1 in the data table) or not (indicated by 0). Very often, attributes are graded (fuzzy) rather than bivalent, i.e. an attribute y applies to an object x to a certain degree. FCA of data tables with fuzzy attributes was studied by several authors, we refer to [7] for the first approach, and to [12] and [1]–[5] for the approach we are using in the present paper.

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The present paper brings up several results related to similarity in FCA of data with fuzzy attributes. Basically, our study is motivated by the following questions: Do similar input data lead to similar outputs of FCA (formal concepts, concept lattices, attribute implications)? Can we obtain estimations of the similarities in question? Can we utilize the similarities to reduce the amount of (input or output) data by putting together similar pieces of data? A study of these problems also tells us about a sensitivity of FCA to exact degrees (in the input data, in the attribute implications, etc.) which is an important issue in fuzzy modeling by itself.

Our paper is a continuation of results from [1] where we studied several issues related to similarity in FCA including a method of parameterized factorization of concept lattices by similarity for which an efficient algorithm was found in [4]. We present results on similarity for concept lattices with hedges [5]. Hedges are parameters controlling the size of concept lattices. Among others, we present formulas for estimation of similarity of concept lattices with hedges in terms of similarity of hedges. Computing the estimations is much faster than computing the exact similarities. Because of the limited scope, we omit proofs and leave them to an extended version of this paper.

2 Preliminaries

We use sets of truth degrees equipped with operations (logical connectives) which form complete residuated lattices, i.e. algebras $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for each $a, b, c \in L$. A truth-stressing hedge (shortly, a hedge) [10, 11] on $\mathbf{L}$ is a unary operation $^* : L \rightarrow L$ satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true" and properties (i)–(iv) have natural interpretations, see [10, 11].

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization: $1^* = 1$, $a^* = 0$ ($a < 1$). Note that a special case of a complete residuated lattice with a hedge is a two-element Boolean algebra of classical (bivalent) logic.

Having $\mathbf{L}$, we define usual notions [2, 10]: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \big(A(u) \to B(u)\big), \tag{1}$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^U$). Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Given $A, B \in \mathbf{L}^U$, we define an equality degree

$$A \approx B = \textstyle\bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big). \tag{2}$$

It is easily seen that $A \approx B = S(A, B) \wedge S(B, A)$.

A fuzzy relation E in U is called reflexive if for each $u \in U$ we have $E(u, u) = 1$; symmetric if for each $u, v \in U$ we have $E(u, v) = E(v, u)$; transitive if for each $u, v, w \in U$ we have $E(u, v) \otimes E(v, w) \leq E(u, w)$. A fuzzy equivalence [13] in U is a fuzzy relation in U which is reflexive, symmetric, and transitive; a fuzzy equivalence E in U for which $E(u, v) = 1$ implies $u = v$ is called a fuzzy equality. We often denote a fuzzy equivalence by $\approx$ and use an infix notation, i.e. we write $(u \approx v)$ instead of $\approx(u, v)$. If a set U is equipped with a fuzzy equality $\approx$ in U, a fuzzy relation $\preceq$ in U is called a fuzzy order in $\langle U, \approx \rangle$ [2, 13] if $\preceq$ is reflexive, transitive, and antisymmetric w.r.t. $\approx$, i.e. for each $u, v \in U$ we have $(u \preceq v) \wedge (v \preceq u) \leq (u \approx v)$, and if $\preceq$ is compatible with $\approx$, i.e. $(u_1 \preceq v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq (u_2 \preceq v_2)$, see [2] for details.

3 Similarity in concept lattices with hedges

3.1 Concept lattices with hedges

Let X and Y be sets of objects and attributes, respectively, $I \in \mathbf{L}^{X \times Y}$ be a fuzzy relation between X and Y with $I(x, y)$ being interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. The triplet $\langle X, Y, I \rangle$ is called a data table with fuzzy attributes.

Let $*_X$ and $*_Y$ be hedges. For $\mathbf{L}$-sets $A \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects), $B \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes) we define $\mathbf{L}$-sets $A^{\uparrow} \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes), $B^{\downarrow} \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects) by $A^{\uparrow}(y) = \bigwedge_{x \in X}\big(A(x)^{*_X} \rightarrow I(x,y)\big)$, and $B^{\downarrow}(x) = \bigwedge_{y \in Y}\big(B(y)^{*_Y} \rightarrow I(x,y)\big)$. We put $\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\, B^{\downarrow} = A\}$. For $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators ${}^{\downarrow}, {}^{\uparrow}$ form a Galois connection with hedges [5]. $\langle \mathcal{B}(X^{*_X}, Y^{*_Y}, I), \leq\rangle$ is called a (fuzzy) concept lattice with hedges ^{*_X} and ^{*_Y} induced by $\langle X, Y, I\rangle$ [5]. For $*_Y = \mathrm{id}_L$ (identity), we write only $\mathcal{B}(X^{*_X}, Y, I)$. Elements $\langle A, B\rangle$ of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. A and B are called the extent and the intent of the concept $\langle A, B\rangle$, respectively, and represent the collection of all objects and all attributes covered by $\langle A, B\rangle$. $\leq$ models a subconcept-superconcept hierarchy.

For each $\langle X, Y, I\rangle$ we consider a set $\mathrm{Ext}(X^{*_X}, Y^{*_Y}, I) \subseteq \mathbf{L}^X$ of all extents and a set $\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) \subseteq \mathbf{L}^Y$ of all intents of concepts of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, i.e.

$$\mathrm{Ext}(X^{*_X}, Y^{*_Y}, I) = \{A \in \mathbf{L}^X \mid \langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \text{ for some } B \in \mathbf{L}^Y\},$$

$$\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) = \{B \in \mathbf{L}^Y \mid \langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \text{ for some } A \in \mathbf{L}^X\}.$$

For details, we refer to [2, 3, 5].

3.2 Similarity of concept lattices with hedges

The aim of this section is to study relationships between $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$ and $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$, i.e. between sets of clusters extracted from a data table $\langle X, Y, I\rangle$ which differ in hedges (^{*_1} and ^{*_3} in case of $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$, and ^{*_2} and ^{*_4} in case of $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$). The reason for this study is the following. Hedges ^{*_X} and ^{*_Y} serve as parameters. Their primary role is to control the size of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, i.e. to control the number $|\mathcal{B}(X^{*_X}, Y^{*_Y}, I)|$ of clusters extracted from data table $\langle X, Y, I\rangle$. Basic theoretical and experimental results were presented in [5]. In particular, it was demonstrated in [5] that tuning the hedges leads to a smooth change of the size of concept lattices, and that stronger hedges lead to a smaller number of extracted formal concepts, see [5] and Section 4. Preliminary theoretical results were also obtained in [5]. For instance, it was shown that if both ^{*_1} and ^{*_2} are identities and if ^{*_3} is stronger than ^{*_4}, i.e. $a^{*_3} \leq a^{*_4}$ for each $a \in L$, then $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$ is a subset of $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$.

First, we need to propose a tractable definition of a degree to which $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$ is similar to $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$. The definitions follow. We start by a degree $\mathcal{M}_1 \preceq \mathcal{M}_2$ to which a system $\mathcal{M}_1$ of fuzzy sets is contained in a system $\mathcal{M}_2$ of fuzzy sets. A degree $\mathcal{M}_1 \approx \mathcal{M}_2$ to which $\mathcal{M}_1$ and $\mathcal{M}_2$ are similar is then defined as a “conjunction” of $\mathcal{M}_1 \preceq \mathcal{M}_2$ and $\mathcal{M}_2 \preceq \mathcal{M}_1$.

Definition 1. *For systems $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbf{L}^U$ of fuzzy sets in U define*

$$\mathcal{M}_1 \preceq \mathcal{M}_2 = \bigwedge_{A_1 \in \mathcal{M}_1} \bigvee_{A_2 \in \mathcal{M}_2} A_1 \approx A_2,$$
$$\mathcal{M}_1 \approx \mathcal{M}_2 = (\mathcal{M}_1 \preceq \mathcal{M}_2) \wedge (\mathcal{M}_2 \preceq \mathcal{M}_1).$$

Remark 1. (1) $A_1 \approx A_2$ is a degree of equality defined by (2). Note that we may take another suitable definition of $A_1 \approx A_2$. In this paper, however, we do not consider other options.

(2) It can be seen that $\mathcal{M}_1 \preceq \mathcal{M}_2$ is a truth degree of "for each $A_1 \in \mathcal{M}_1$ there is $A_2 \in \mathcal{M}_2$ such that A_1 and A_2 are similar."

The following is another definition which formalizes the same idea.

Definition 2. *For systems $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbf{L}^U$ of fuzzy sets in U define*

$$\mathcal{M}_1 \preceq_c \mathcal{M}_2 = \bigvee\{c \in L \mid \text{ for each } A_1 \in \mathcal{M}_1 \text{ there is } A_2 \in \mathcal{M}_2 : c \leq (A_1 \approx A_2)\},$$
$$\mathcal{M}_1 \approx_c \mathcal{M}_2 = (\mathcal{M}_1 \preceq_c \mathcal{M}_2) \wedge (\mathcal{M}_2 \preceq_c \mathcal{M}_1).$$

Remark 2. Thus, $\mathcal{M}_1 \preceq_c \mathcal{M}_2$ can be seen as the largest degree c such that for each $A_1 \in \mathcal{M}_1$ there is $A_2 \in \mathcal{M}_2$ with $c \leq (A_1 \approx A_2)$.

The following lemma summarizes basic properties of the above concepts.

Lemma 1. *$\approx$ and $\approx_c$ are fuzzy equalities in $2^{\mathbf{L}^U}$. $\preceq$ and $\preceq_c$ are fuzzy orders in $\langle 2^{\mathbf{L}^U}, \approx\rangle$ and $\langle 2^{\mathbf{L}^U}, \approx_c\rangle$. We have $(\mathcal{M}_1 \preceq_c \mathcal{M}_2) \leq (\mathcal{M}_1 \preceq \mathcal{M}_2)$ and $(\mathcal{M}_1 \approx_c \mathcal{M}_2) \leq (\mathcal{M}_1 \approx \mathcal{M}_2)$. If $\mathbf{L}$ is a finite chain then $\preceq$ coincides with $\preceq_c$ and $\approx$ coincides with $\approx_c$.*

Using the above definition, we are going to define degrees of containment and similarity between concept lattices with hedges. Because of the limited scope we present only definitions and results based on $\preceq$ and $\approx$. For the sake of brevity, we denote

$$\mathcal{B}_{1,3} = \mathcal{B}(X^{*_1}, Y^{*_3}, I) \quad \text{and} \quad \mathcal{B}_{2,4} = \mathcal{B}(X^{*_2}, Y^{*_4}, I).$$

Definition 3. *Put*

$$\mathcal{B}_{1,3} \preceq_{\mathrm{Ext}} \mathcal{B}_{2,4} = \mathrm{Ext}(X^{*_1}, Y^{*_3}, I) \preceq \mathrm{Ext}(X^{*_2}, Y^{*_4}, I),$$
$$\mathcal{B}_{1,3} \preceq_{\mathrm{Int}} \mathcal{B}_{2,4} = \mathrm{Int}(X^{*_1}, Y^{*_3}, I) \preceq \mathrm{Int}(X^{*_2}, Y^{*_4}, I),$$
$$\mathcal{B}_{1,3} \approx_{\mathrm{Ext}} \mathcal{B}_{2,4} = \mathrm{Ext}(X^{*_1}, Y^{*_3}, I) \approx \mathrm{Ext}(X^{*_2}, Y^{*_4}, I),$$
$$\mathcal{B}_{1,3} \approx_{\mathrm{Int}} \mathcal{B}_{2,4} = \mathrm{Int}(X^{*_1}, Y^{*_3}, I) \approx \mathrm{Int}(X^{*_2}, Y^{*_4}, I).$$

Remark 3. (1) Thus, $\mathcal{B}_{1,3} \preceq_{\mathrm{Ext}} \mathcal{B}_{2,4}$ is a degree to which the system of extents of $\mathcal{B}_{1,3}$ is contained in the system of extents of $\mathcal{B}_{2,4}$ in the sense of Definition 2; analogously for $\mathcal{B}_{1,3} \preceq_{\mathrm{Int}} \mathcal{B}_{2,4}$, $\mathcal{B}_{1,3} \approx_{\mathrm{Ext}} \mathcal{B}_{2,4}$, and $\mathcal{B}_{1,3} \approx_{\mathrm{Int}} \mathcal{B}_{2,4}$. Note that in general, $\mathcal{B}_{1,3} \preceq_{\mathrm{Ext}} \mathcal{B}_{2,4}$ may differ from $\mathcal{B}_{1,3} \preceq_{\mathrm{Int}} \mathcal{B}_{2,4}$ and the choice is up to the user (whether he/she is interested in clusters of objects or attributes). One might of course take also $\mathcal{B}_{1,3} \preceq \mathcal{B}_{2,4} = (\mathcal{B}_{1,3} \preceq_{\text{Ext}} \mathcal{B}_{2,4}) \wedge (\mathcal{B}_{1,3} \preceq_{\text{Int}} \mathcal{B}_{2,4})$.

(2) Note that we have $\mathcal{B}_{1,3} \approx_{\text{Ext}} \mathcal{B}_{2,4} = (\mathcal{B}_{1,3} \preceq_{\text{Ext}} \mathcal{B}_{2,4}) \wedge (\mathcal{B}_{2,4} \preceq_{\text{Ext}} \mathcal{B}_{1,3})$ and $\mathcal{B}_{1,3} \approx_{\text{Int}} \mathcal{B}_{2,4} = (\mathcal{B}_{1,3} \preceq_{\text{Int}} \mathcal{B}_{2,4}) \wedge (\mathcal{B}_{2,4} \preceq_{\text{Int}} \mathcal{B}_{1,3})$.

We are going to present formulas for estimation of lower bounds of the degrees introduced in Definition 3. This is particularly interesting if we start with $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$, find out that $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$ is too large, replace hedges ^{*_1} and ^{*_3} by (stronger) hedges ^{*_2} and ^{*_4}, and consider $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$ instead of $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$. Then one wants to know a degree to which $\mathcal{B}(X^{*_2}, Y^{*_4}, I)$ is contained in (similar to) $\mathcal{B}(X^{*_1}, Y^{*_3}, I)$. We need the following definition describing relationships between hedges.

Definition 4. *For hedges $*_1$ and $*_2$ on $\mathbf{L}$ put*

$$(*_1 \preceq *_2) = \textstyle\bigwedge_{a \in L}(a^{*_1} \to a^{*_2}),$$
$$(*_1 \approx *_2) = \textstyle\bigwedge_{a \in L}(a^{*_1} \leftrightarrow a^{*_2}).$$

Remark 4. (1) Since $a \leftrightarrow b$ (degree of equivalence of a and b) can be seen as a degree to which degrees a and b are similar, $*_1 \approx *_2$ can be interpreted as a degree to which hedges $*_1$ and $*_2$ are similar (yield similar results). Analogously, $*_1 \preceq *_2$ can be interpreted as a degree to which $*_1$ is stronger than $*_2$.

(2) Note that $(*_1 \approx *_2) = (*_1 \preceq *_2) \wedge (*_2 \preceq *_1)$.

Lemma 2. *$\approx$ is a fuzzy equality relation on the set of all truth stressers on $\mathbf{L}$; $\preceq$ is a fuzzy order on the set of all truth stressers on $\mathbf{L}$ equipped with $\approx$.*

Degrees $*_1 \preceq *_2$, $*_1 \approx *_2$ enable us to deduce some natural properties of hedges $*_1$ and $*_2$. As an example, for a hedge $*$ on $\mathbf{L}$ denote

$$\text{fix}(*) = \{a \in L \mid a^* = a\}$$

the set of all fixpoints of $*$. For $K_1, K_2 \subseteq L$, put

$$(K_1 \preceq K_2) = \textstyle\bigwedge_{a \in K_1} \bigvee_{b \in K_2}(a \leftrightarrow b),$$
$$(K_1 \approx K_2) = (K_1 \preceq K_2) \wedge (K_2 \preceq K_1).$$

$K_1 \preceq K_2$ ($K_1 \approx K_2$) is a degree to which K_1 is contained in (similar to) K_2. The following lemma shows that stronger hedges have smaller sets of fixpoints and that similar hedges have similar sets of fixpoints.

Lemma 3. $(*_1 \preceq *_2) \leq (\text{fix}(*_1) \preceq \text{fix}(*_2))$, $(*_1 \approx *_2) \leq (\text{fix}(*_1) \approx \text{fix}(*_2))$.

Before presenting main results of this section, we need some auxiliary claims. For $i = 1, 2, 3, 4$, denote by ${}^{\uparrow_i}$ and ${}^{\downarrow_i}$ the mappings induced by a hedge ^{*_i}, i.e. $A^{\uparrow_i}(y) = \bigwedge_{x \in X}(A^{*_i}(x) \to I(x, y))$ and $B^{\downarrow_i}(x) = \bigwedge_{y \in Y}(B^{*_i}(y) \to I(x, y))$. Then we have

Lemma 4. *For $A \in \mathbf{L}^X$, $(*_1 \preceq *_3) \leq S(A^{\uparrow_3}, A^{\uparrow_1})$ and $(*_1 \approx *_3) \leq (A^{\uparrow_1} \approx A^{\uparrow_3})$; dually for $B \in \mathbf{L}^Y$ and $^{\downarrow_2}$ and $^{\downarrow_4}$.*

The main results providing estimations of degrees of containment and degrees of similarity of concept lattices with hedges follow.

Theorem 1. *We have the following estimation formulas:*

$$
\begin{aligned}
(*_1 \preceq *_2) \otimes (*_3 \approx *_4) &\leq \mathcal{B}_{1,3} \preceq_{\mathrm{Ext}} \mathcal{B}_{2,4},\\
(*_3 \preceq *_4) \otimes (*_1 \approx *_2) &\leq \mathcal{B}_{1,3} \preceq_{\mathrm{Int}} \mathcal{B}_{2,4},\\
(*_1 \approx *_2) \otimes (*_3 \approx *_4) &\leq \mathcal{B}_{1,3} \approx_{\mathrm{Ext}} \mathcal{B}_{2,4},\\
(*_1 \approx *_2) \otimes (*_3 \approx *_4) &\leq \mathcal{B}_{1,3} \approx_{\mathrm{Int}} \mathcal{B}_{2,4}.
\end{aligned}
$$

The next two theorems provide formulas for the case when the hedges by attributes are equal and, moreover, are identities.

Theorem 2. *We have the following estimation formulas:*

$$
\begin{aligned}
(*_1 \preceq *_2) &\leq \mathcal{B}(X^{*_1}, Y^{*_Y}, I) \preceq_{\mathrm{Ext}} \mathcal{B}(X^{*_2}, Y^{*_Y}, I),\\
(*_1 \approx *_2) &\leq \mathcal{B}(X^{*_1}, Y^{*_Y}, I) \preceq_{\mathrm{Int}} \mathcal{B}(X^{*_2}, Y^{*_Y}, I),\\
(*_1 \approx *_2) &\leq \mathcal{B}(X^{*_1}, Y^{*_Y}, I) \approx_{\mathrm{Ext}} \mathcal{B}(X^{*_2}, Y^{*_Y}, I),\\
(*_1 \approx *_2) &\leq \mathcal{B}(X^{*_1}, Y^{*_Y}, I) \approx_{\mathrm{Int}} \mathcal{B}(X^{*_2}, Y^{*_Y}, I).
\end{aligned}
$$

Theorem 3. *For $*_Y$ being identity on L we have*

$$(*_1 \preceq *_2) \leq \mathcal{B}(X^{*_1}, Y^{*_Y}, I) \preceq_{\mathrm{Int}} \mathcal{B}(X^{*_2}, Y^{*_Y}, I).$$

Remark 5. Note that for ^{*_Y} being identity we not only have the previous better estimation but one can show that for $\langle A, B\rangle \in \mathcal{B}(X^{*_1}, Y^{*_Y}, I)$ there is $\langle C, D\rangle \in \mathcal{B}(X^{*_2}, Y^{*_Y}, I)$ such that both $(*_1 \preceq *_2) \leq (A \approx C)$ and $(*_1 \preceq *_2) \leq (B \approx D)$. Note also that the case when ^{*_X} is identity (dual situation to ^{*_Y} being identity) is important in studying fuzzy attribute implications.

We postpone further ramifications of the previous results to an extended version of this paper.

4 Example and remarks

This section presents an illustrative example of similarities of concept lattices with hedges. Let $\mathbf{L}$ be a finite Łukasiewicz chain with $L = \{0, 0.25, 0.5, 0.75, 1\}$. There are five hedges on $\mathbf{L}$; they are depicted in Fig. 1: the left-most hedge (denoted by $*_1$) is globalization on L, the right-most one (denoted by $*_5$) is identity on L; for $*_2$ we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, $1^{*_2} = 1$, etc. Table 1 (left) contains the fuzzy order $\preceq$ on hedges: a table

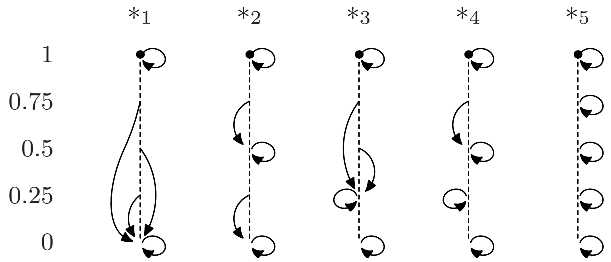

Fig. 1. Truth stressers of five-element Łukasiewicz chain

Table 1. Fuzzy order and fuzzy equality on truth-stressers

$\preceq$	$*_1$	$*_2$	$*_3$	$*_4$	$*_5$
$*_1$	1	1	1	1	1
$*_2$	0.5	1	0.75	1	1
$*_3$	0.75	0.75	1	1	1
$*_4$	0.5	0.75	0.75	1	1
$*_5$	0.25	0.75	0.5	0.75	1

$\approx$	$*_1$	$*_2$	$*_3$	$*_4$	$*_5$
$*_1$	1	0.5	0.75	0.5	0.25
$*_2$	0.5	1	0.75	0.75	0.75
$*_3$	0.75	0.75	1	0.75	0.5
$*_4$	0.5	0.75	0.75	1	0.75
$*_5$	0.25	0.75	0.5	0.75	1

entry corresponding to a row $*_i$ and a column $*_j$ contains degree $*_i \preceq *_j$. Table 1 (right) displays the fuzzy equality $\approx$ on hedges.

Consider a data table $\langle X, Y, I \rangle$ given by table in Fig. 2 (left). The set X of object consists of "Mercury", "Venus", ..., set Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). Since we have five hedges on $\mathbf{L}$, the input data table $\langle X, Y, I \rangle$ induces 25 (possibly different) output concept lattices with hedges $\mathcal{B}(X^{*_1}, Y^{*_1}, I), \ldots, \mathcal{B}(X^{*_5}, Y^{*_5}, I)$. For brevity, each concept lattice $\mathcal{B}(X^{*_i}, Y^{*_j}, I)$ will be denoted by $\mathcal{B}_{i,j}$. All the concept lattices are depicted in Fig. 2 (right) ($\mathcal{B}_{i,j}$ lies on the intersection of row $*_i$ and column $*_j$). Each concept lattice is depicted by its Hasse diagram. The nodes of the diagram represent the clusters (concepts) in the data $\langle X, Y, I \rangle$; the edges represent the partial ordering of the clusters (subconcept-superconcept hierarchy), see Section 3.1. Note that $\mathcal{B}_{1,1}$ (both hedges are globalization) contains 8 clusters (concepts) while $\mathcal{B}_{5,5}$ (both hedges are identity) consists of 216 clusters.

Looking at the concept lattices in Fig. 2, we might say that, e.g., $\mathcal{B}_{4,5}$, $\mathcal{B}_{5,4}$, and $\mathcal{B}_{5,5}$ are (very) similar while $\mathcal{B}_{1,1}$ is quite different (much simpler) from each of $\mathcal{B}_{4,5}$, $\mathcal{B}_{5,4}$, and $\mathcal{B}_{5,5}$. This intuitive observation agrees with degrees of similarity of these fuzzy concept lattices. Indeed, Table 2 (left) contains degrees of similarity of intents, i.e. a table entry on the intersection of row $\mathcal{B}_{i,k}$ and column $\mathcal{B}_{j,l}$ contains degree $\mathcal{B}_{i,k} \approx_{\mathrm{Int}} \mathcal{B}_{j,l}$. We have, e.g., $\mathcal{B}_{4,5} \approx_{\mathrm{Int}} \mathcal{B}_{5,5} = 0.75$ while $\mathcal{B}_{1,1} \approx_{\mathrm{Int}} \mathcal{B}_{5,5} = 0.25$. Table 2 (right) contains estimations of degrees of similarity of intents: a table entry on the intersection of row $\mathcal{B}_{i,k}$ and column $\mathcal{B}_{j,l}$ contains truth degree $(*_i \approx *_j) \otimes (*_k \approx *_l)$. Observe that in some cases, estimations given by Table 2 (right) are equal to the values of

	size		distance	
	small	large	far	near
Mercury	1	0	0	1
Venus	0.75	0	0	1
Earth	0.75	0	0	0.75
Mars	1	0	0.5	0.75
Jupiter	0	1	0.75	0.5
Saturn	0	1	0.75	0.5
Uranus	0.25	0.5	1	0.25
Neptune	0.25	0.5	1	0
Pluto	1	0	1	0

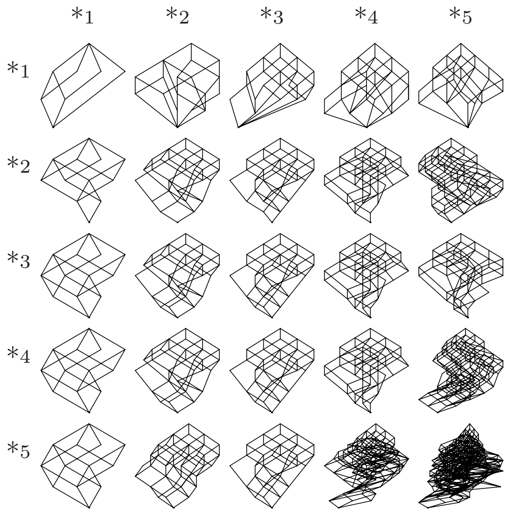

Fig. 2. Data table with fuzzy attributes (left); concept lattices generated from the data table by all combinations of truth stressers $*_X$ and $*_Y$ from Fig. 1 (right).

similarity over intents, in some cases, however, the estimation is strictly lower. On the other hand, the cost of computing values $(*_i \approx *_j) \otimes (*_k \approx *_l)$ is much smaller than the cost of computing $\mathcal{B}_{i,k} \approx_{\mathrm{Int}} \mathcal{B}_{j,l}$ especially in case of large input data. Table 3 depicts fuzzy order over extents, i.e. $\mathcal{B}_{i,k} \preceq_{\mathrm{Ext}} \mathcal{B}_{j,l}$, and its estimation, i.e. $(*_i \preceq *_j) \otimes (*_k \approx *_l)$. Note that the estimations of Theorems 2 and 3 provide closer approximations of the estimated degrees (details in the extended version).

Table 2. Similarity over intents and its estimation

$\approx_{\mathrm{Int}}$	$\mathcal{B}_{1,1}$	$\mathcal{B}_{3,3}$	$\mathcal{B}_{4,5}$	$\mathcal{B}_{5,4}$	$\mathcal{B}_{5,5}$
$\mathcal{B}_{1,1}$	1	0.5	0.5	0.5	0.25
$\mathcal{B}_{3,3}$	0.5	1	0.5	0.5	0.5
$\mathcal{B}_{4,5}$	0.5	0.5	1	0.75	0.75
$\mathcal{B}_{5,4}$	0.5	0.5	0.75	1	0.75
$\mathcal{B}_{5,5}$	0.25	0.5	0.75	0.75	1

est.	$\mathcal{B}_{1,1}$	$\mathcal{B}_{3,3}$	$\mathcal{B}_{4,5}$	$\mathcal{B}_{5,4}$	$\mathcal{B}_{5,5}$
$\mathcal{B}_{1,1}$	1	0.5	0	0	0
$\mathcal{B}_{3,3}$	0.5	1	0.25	0.25	0
$\mathcal{B}_{4,5}$	0	0.25	1	0.5	0.75
$\mathcal{B}_{5,4}$	0	0.25	0.5	1	0.75
$\mathcal{B}_{5,5}$	0	0	0.75	0.75	1

5 Future research

Future research as well as topics which did not fit the limited extent of this paper include the following: factorization of concept lattices with hedges and collections of attribute implications by putting together similar concepts and

Table 3. Fuzzy order over extents and its estimation

$\preceq_{\mathrm{Ext}}$	$\mathcal{B}_{1,1}$	$\mathcal{B}_{3,3}$	$\mathcal{B}_{4,5}$	$\mathcal{B}_{5,4}$	$\mathcal{B}_{5,5}$
$\mathcal{B}_{1,1}$	1	1	1	1	1
$\mathcal{B}_{3,3}$	0.5	1	0.75	1	1
$\mathcal{B}_{4,5}$	0.5	0.75	1	0.75	1
$\mathcal{B}_{5,4}$	0.5	0.75	0.75	1	1
$\mathcal{B}_{5,5}$	0.25	0.75	0.75	0.75	1

est.	$\mathcal{B}_{1,1}$	$\mathcal{B}_{3,3}$	$\mathcal{B}_{4,5}$	$\mathcal{B}_{5,4}$	$\mathcal{B}_{5,5}$
$\mathcal{B}_{1,1}$	1	0.75	0.25	0.5	0.25
$\mathcal{B}_{3,3}$	0.5	1	0.5	0.75	0.5
$\mathcal{B}_{4,5}$	0	0.25	1	0.75	1
$\mathcal{B}_{5,4}$	0	0.25	0.5	1	0.75
$\mathcal{B}_{5,5}$	0	0	0.75	0.75	1

similar implications; results concerning similarity and validity of attribute implications; similarity of theories consisting of attribute implications [6] (do similar data tables have similar non-redundant bases of attribute implications? etc.); similarity results based on other measures of similarity of fuzzy sets.

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