Continuous fuzzy Horn logic

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The paper deals with fuzzy Horn logic (FHL) which is a fragment of predicate fuzzy logic with evaluated syntax. Formulas of FHL are of the form of simple implications between identities. We show that one can have Pavelka-style completeness of FHL w. r. t. semantics over the unit interval $[0, 1]$ with (residuated lattices given by) left-continuous t-norm and a residuated implication, provided that only certain fuzzy sets of formulas are considered. The model classes of fuzzy structures of FHL are characterized by closure properties. We also give comments on related topics proposed by N. Weaver.

1 Introduction

Fuzzy logic in narrow sense (formal fuzzy logic or mathematical fuzzy logic) has been substantially developed in the past few decades. Fundamental contributions to fuzzy logic in narrow sense are due to Hájek [14] (logic with syntax in the classical style), and Pavelka and Novák [16, 15] (logic with evaluated syntax). It is almost a matter of folklore that these two basic approaches to fuzzy logic are now called *Hájek-style fuzzy logic* and *Pavelka-style fuzzy logic*. The present paper, which is a continuation of [5, 6], deals with so-called *fuzzy Horn logic* (FHL) which is basically an equational fragment of predicate fuzzy logic with evaluated syntax. Our motivation is the following. The basic principle of fuzzy logic with evaluated syntax (Pavelka-style logic) is that one works with degrees of semantic consequence and degrees of provability. A fuzzy logical calculus developed in Pavelka-style is called *Pavelka-complete* if the degrees of semantic consequence agree with the degrees of provability. In the early results [16], Pavelka showed an essential limitation of logics with evaluated syntax: the standard Łukasiewicz algebra is among all the interesting structures of truth degrees defined on the real unit interval the only one for which we can have a Pavelka-style complete propositional fuzzy logic with implication interpreted by a residuum. This observation applies to the predicate case as well, see also [10, 15]. The limitation opens a question, under which conditions we can have a Pavelka-style completeness if we restrict ourselves only to a small subset of relatively simple formulas. In this paper we deal with formulas which are of the form of implications between (truth-weighted) identities. Such formulas have already been introduced in [5], where we showed that the complexity of premises of the formulas in question influences the classes of structures of truth degrees for which we can prove Pavelka-style completeness of FHL. We are now going to show that, considering only the systems of formulas which fulfil certain type of continuity, fuzzy Horn logic as introduced in [5] is Pavelka-complete for all structures of truth degrees given by the unit interval $[0, 1]$ and a left-continuous t-norm (MTL-algebras on $[0, 1]$, see [9]). Thus, from the point of view of fuzzy logic with evaluated syntax, we are going to show that the continuity of operations of structures of truth degrees [15, 16] is not necessary to prove completeness provided that we restrict ourselves only to certain formulas and (continuous) theories.

We are also interested in characterizing the model classes of fuzzy structures by closure properties. Since we work with predicate languages containing a single relation symbol, namely the symbol $\approx$ for equality, the models for our languages are the so-called *algebras with fuzzy equalities* [4]: an algebra with fuzzy equality

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is a particular fuzzy structure with a functional part (classical algebra) endowed with a similarity (a particular fuzzy equivalence relation) such that each function is in an appropriate sense compatible with the similarity. Compatibility, which is a semantic translation of the classical compatibility (congruence) axiom

(1) $$x_1 \approx y_1 \,\&\cdots\&\, x_n \approx y_n \Rightarrow f(x_1,\dots,x_n) \approx f(y_1,\dots,y_n)$$

ensures that each function of an algebra with fuzzy equality yields similar results if applied to pairwise similar arguments. Thus, algebras with fuzzy equalities can be seen as classical algebras with constrained operations. The present paper shows that classes of algebras with fuzzy equalities closed under certain closure operators are exactly the model classes of FHL which, in a sense, satisfy all formulas continuously. Interestingly enough, an analogous idea was introduced by N. Weaver in [20] who studied quasivarieties of the so-called metric algebras. Note that metric algebras can also be seen as algebras with constrained operations. In this case, the constraint is formulated in terms of a metric and equicontinuous satisfaction. Since the idea behind algebras with fuzzy equalities and metric algebras is similar in sense that both represent structures mapping "similar (close) elements" to "similar (close) results", we also present a comparison of both approaches.

In [18], the author presents a generalization of logic programming which is shown to be Pavelka-style complete over structures of truth degrees defined on $[0,1]$. IF-THEN rules used in [18] can be seen as a generalization of Horn clauses used in the present paper, however, there are important differences between both calculi. For instance, analogously as the classical logic programming, fuzzy logic programming deals with finite theories (generalizations of definite programs) while we allow for infinite theories. Infinite theories are essential because we wish to deal with definability of classes of (fuzzy) structures by collections of formulas. A comparison of fuzzy logic programming in sense of [18] and FHL will be the subject of a forthcoming paper.

Our paper is organized as follows. In Section 2, we briefly summarize preliminary notions. Section 3 introduces algebras with fuzzy equalities and basic syntactic and semantic notions of FHL. In Section 4, we describe metric algebras. In Section 5, we discuss the relationship between algebras with fuzzy equalities and metric algebras. Section 6 contains results on characterization of model classes of algebras with fuzzy equalities by closure properties. Finally, in Section 7 we present the completeness result.

2 Preliminaries

We use complete residuated lattices as the basic structures of truth degrees. Residuated lattices, being introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [11, 12]. A thorough information about the role of residuated lattices in fuzzy logic and fuzzy relational systems can be obtained from monographs [2, 13, 14]. Recall that a (complete) residuated lattice is an algebra

$$\boldsymbol{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

of type $\langle 2,2,2,2,0,0\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1;

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid;

(iii) $\otimes, \rightarrow$ form an adjoint pair, i. e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$.

The most studied and applied complete residuated lattices are those defined on the real interval $[0,1]$ with $\wedge$ and $\vee$ being the minimum and the maximum, respectively. Such residuated lattices uniquely correspond to left-continuous t-norms. In more detail,

$$\boldsymbol{L} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$$

is a complete residuated lattice iff $\otimes$ is a left-continuous t-norm (i. e. $\otimes$ is left-continuous, commutative, associative, monotone, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in [0,1]$) and the residuum $\rightarrow$ is given by

$$a \rightarrow b = \bigvee\{c \mid a \otimes c \leq b\}.$$

Residuated lattices on $[0,1]$ given by left-continuous (continuous) t-norms are exactly the MTL-algebras (BL-algebras) defined on $[0,1]$ with $\wedge = \min$ and $\vee = \max$, see [2, 9, 14]. A left-continuous t-norm $\otimes$ is continuous iff

$$a \otimes \textstyle\bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \otimes b_i).$$

A continuous t-norm $\otimes$ is called *Archimedean* if $0, 1$ are its only idempotents (i. e. $a \otimes a < a$ for any $0 < a < 1$).

A continuous Archimedean t-norm $\otimes$ is called *strict* if 0 is its only nilpotent element (i. e. $a \otimes \cdots \otimes a > 0$ for any $a > 0$). Three most important pairs of adjoint operations given by continuous t-norms are:

1. Łukasiewicz:

$$a \otimes b = \max(a + b - 1, 0), \quad a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a + b & \text{otherwise;} \end{cases}$$

2. Gödel (minimum):

$$a \otimes b = \min(a, b), \quad a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise;} \end{cases}$$

3. Goguen (product):

$$a \otimes b = a \cdot b, \quad a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{otherwise.} \end{cases}$$

In what follows we will use basic properties of residuated lattices given by left-continuous (continuous) t-norms which can be found in [2, 9, 10, 13, 14].

An ***L**-set* A (or *fuzzy set with truth degrees in complete residuated lattice* $\boldsymbol{L}$) *in a universe set* U is a mapping $A : U \longrightarrow L$, $A(\mathfrak{u}) \in L$ being interpreted as the truth degree of "element $\mathfrak{u}$ belongs to A". An $\boldsymbol{L}$-set A in U is called *finite* if there are only finitely many elements $\mathfrak{u} \in U$ with $A(\mathfrak{u}) > 0$. A *binary **L**-relation* R *on* U is an $\boldsymbol{L}$-set in the universe set $U \times U$, i. e. it is a mapping $R : U \times U \longrightarrow L$.

3 Algebras with fuzzy equalities

In this section we briefly describe the notion of an algebra with fuzzy equality. Algebras with fuzzy equalities, introduced in [4], are the semantic structures of the equational fragment of fuzzy logic. The initial results on algebras with fuzzy equalities [1, 3] showed their nice logico-algebraic properties. Namely, in [1] the author presented a syntactico-semantically complete calculus for reasoning with fuzzy sets of equalities while [3] showed an analogy of the well-known Birkhoff's variety theorem – varieties of algebras with fuzzy equalities are the model classes of fuzzy sets of identities. These results were generalized further in [5, 6]. Algebras with fuzzy equalities can also be seen as fuzzy structures equipped with functions which map pairwise similar arguments to similar results. Recall that, loosely speaking, an algebra with fuzzy equality is a set with functions on it that is equipped with similarity $\approx$ (a particular fuzzy equivalence relation) such that each function f is in an appropriate sense compatible with $\approx$. Before we introduce the very notion of an algebra with fuzzy equality, we need some more preliminary notions.

An ***L**-equivalence* (*fuzzy equivalence, similarity*) *relation* $\approx^M$ *on* M is a binary $\boldsymbol{L}$-relation on M satisfying

(i) $\mathfrak{a} \approx^M \mathfrak{a} = 1$ (reflexivity),

(ii) $\mathfrak{a} \approx^M \mathfrak{b} = \mathfrak{b} \approx^M \mathfrak{a}$ (symmetry),

(iii) $\mathfrak{a} \approx^M \mathfrak{b} \otimes \mathfrak{b} \approx^M \mathfrak{c} \leq \mathfrak{a} \approx^M \mathfrak{c}$ (transitivity)

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in M$. An $\boldsymbol{L}$-equivalence on U where $\mathfrak{a} \approx^M \mathfrak{b} = 1$ iff $\mathfrak{a} = \mathfrak{b}$ is called an ***L**-equality* (*fuzzy equality*). Given an $\boldsymbol{L}$-equality $\approx^M$, $\mathfrak{a} \approx^M \mathfrak{b}$ can be interpreted as "the degree to which $\mathfrak{a}$ and $\mathfrak{b}$ are similar ($\boldsymbol{L}$-equal)". A couple $\langle M, \approx^M \rangle$, where $M \neq \emptyset$ and $\approx^M$ is an $\boldsymbol{L}$-equality on M, will be called an ***L**-similarity space.* Recall that similarity relations and similarity spaces have been the subject of profound study in fuzzy relational systems [2]. A mapping $f : M^n \longrightarrow M$, where $n \in \mathbb{N}$, is *compatible with a binary **L**-relation* R *on* U if for any $\mathfrak{a}_1, \mathfrak{b}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}_n \in U$ we have

$$R(\mathfrak{a}_1, \mathfrak{b}_1) \otimes \cdots \otimes R(\mathfrak{a}_n, \mathfrak{b}_n) \leq R(f(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), f(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)).$$

Compatibility, being the semantic representation of compatibility (congruence) axiom (1), has a natural verbal description: it says "if $\mathfrak{a}_1$ and $\mathfrak{b}_1$ are R-related and … and $\mathfrak{a}_n$ and $\mathfrak{b}_n$ are R-related, then $f(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ and $f(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ are R-related". Thus, if R is an $\boldsymbol{L}$-equality relation, mapping f compatible with R can be seen as a mapping sending pairwise similar arguments to similar results considering the interpretation of "being similar" to be given by the $\boldsymbol{L}$-equality relation R.

We now introduce the notion of an algebra with fuzzy equality. As usual, by a *type* we mean a collection F of function symbols $f \in F$ together with their arities (since we mostly work with a fixed type, we will not mention it explicitly). An *algebra with $\boldsymbol{L}$-equality* (shortly an *$\boldsymbol{L}$-algebra*) *of type* F is a triplet $\boldsymbol{M} = \langle M, \approx^{\boldsymbol{M}}, F^{\boldsymbol{M}} \rangle$, where

(i) $\langle M, F^{\boldsymbol{M}} \rangle$ is a classical algebra of type F,

(ii) $\langle M, \approx^{\boldsymbol{M}} \rangle$ is an $\boldsymbol{L}$-similarity space,

(iii) each function $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$ is compatible with $\approx^{\boldsymbol{M}}$.

Let us now summarize basic syntactic and semantic notions of FHL that will be used in further sections. In the sequel, $t, s, \ldots$ and $t \approx t', s \approx s', \ldots$ denote terms (defined as usual) and identities (of a given type F), respectively. The set of all terms of type F in variables X will be denoted by $T(X)$. Until further notice, we assume that X is a denumerable set of (object) variables. A *Horn clause* (*with truth-weighted premises*) is a syntactic expression of the form

$$(2) \qquad \langle s_1 \approx s_1', a_1 \rangle \,\&\, \cdots \,\&\, \langle s_n \approx s_n', a_n \rangle \Rightarrow (t \approx t'),$$

where $s_1 \approx s_1', \ldots, s_n \approx s_n'$ are pairwise distinct identities and $a_i \in L$ $(i = 1, \ldots, n)$. For brevity, we denote (2) by $P \Rightarrow (t \approx t')$, where P, called *$\boldsymbol{L}$-set of premises*, is the finite binary $\boldsymbol{L}$-relation on $T(X)$ defined by

$$P(r, r') = \begin{cases} a_i & \text{if } r = s_i \text{ and } r' = s_i', \\ 0 & \text{otherwise.} \end{cases}$$

Horn clauses, being a particular type of more general formulas introduced in [5], are the basic formulas of FHL. Note that $P(s, s') \in L$ can be interpreted as the degree (weight) to which identity $s \approx s'$ belongs to P. The intended meaning of (2), abbreviated by $P \Rightarrow (t \approx t')$, is: "If s_1 equals s_1' in degree (at least) $P(s_1, s_1') = a_1$, and … and s_n equals s_n' in degree (at least) $P(s_n, s_n') = a_n$, then t equals t'". Note that each Horn clause of the form (2) can be understood as a shorthand for an ordinary formula of fuzzy predicate logic (with constants of truth degrees in language), see [5].

The interpretation $\|t\|_{\boldsymbol{M},v}$ of a term t in an $\boldsymbol{L}$-algebra $\boldsymbol{M}$ under a valuation $v : X \longrightarrow M$ (of object variables from X) is defined as usual, i. e. $\|t\|_{\boldsymbol{M},v} = v(x)$ if t is a variable x, and $\|t\|_{\boldsymbol{M},v} = f^{\boldsymbol{M}}(\|t_1\|_{\boldsymbol{M},v}, \ldots, \|t_n\|_{\boldsymbol{M},v})$ if t is of the form $f(t_1, \ldots, t_n)$. For an identity $t \approx t'$ we define degree $\|t \approx t'\|_{\boldsymbol{M},v}$ to which $t \approx t'$ is true in $\boldsymbol{M}$ under v by putting $\|t \approx t'\|_{\boldsymbol{M},v} = \|t\|_{\boldsymbol{M},v} \approx^{\boldsymbol{M}} \|t'\|_{\boldsymbol{M},v}$. For each Horn clause $P \Rightarrow (t \approx t')$ we define degree $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},v}$ to which $P \Rightarrow (t \approx t')$ is true in $\boldsymbol{M}$ under v by

$$\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},v} = \begin{cases} \|t \approx t'\|_{\boldsymbol{M},v} & \text{if } P(s, s') \leq \|s \approx s'\|_{\boldsymbol{M},v} \text{ for all } s, s' \in T(X), \\ 1 & \text{otherwise.} \end{cases}$$

The interpretation of Horn clauses as defined above is not ad hoc in the sense that it can be equivalently defined as an interpretation[1)] of an ordinary formula (with constants for truth degrees) which corresponds to $P \Rightarrow (t \approx t')$.

Since we are interested in classes of $\boldsymbol{L}$-algebras given by Horn clauses, we define notions of Horn classes and Horn theories. Let Σ be an $\boldsymbol{L}$-set of Horn clauses. An $\boldsymbol{L}$-algebra $\boldsymbol{M}$ is called a *model of* Σ if

$$\Sigma(P \Rightarrow (t \approx t')) \leq \|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},v}$$

holds for any Horn clause $P \Rightarrow (t \approx t')$ and any valuation $v : X \longrightarrow M$. The class of all models of Σ, denoted by $\mathrm{Mod}(\Sigma)$, is called *the Horn class of* Σ. For each class $\mathcal{K}$ of $\boldsymbol{L}$-algebras we consider an $\boldsymbol{L}$-set $\mathrm{Horn}(\mathcal{K})$ of Horn clauses such that

$$(3) \qquad (\mathrm{Horn}(\mathcal{K}))(P \Rightarrow (t \approx t')) = \bigwedge \{\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},v} \mid \boldsymbol{M} \in \mathcal{K} \text{ and } v : X \longrightarrow M\}.$$

$\mathrm{Horn}(\mathcal{K})$ is called *the Horn theory of* $\mathcal{K}$. Loosely speaking, $\mathrm{Mod}(\Sigma)$ is the class of all $\boldsymbol{L}$-algebras satisfying Σ, and $\mathrm{Horn}(\mathcal{K})$ is the $\boldsymbol{L}$-set of all Horn clauses valid in $\mathcal{K}$, respectively. Having defined $\mathrm{Mod}(\Sigma)$ and $\mathrm{Horn}(\mathcal{K})$, we can introduce a *degree of semantic consequence*: degree $\|P \Rightarrow (t \approx t')\|_\Sigma$ to which $P \Rightarrow (t \approx t')$ follows semantically from Σ is defined by putting $\|P \Rightarrow (t \approx t')\|_\Sigma = (\mathrm{Horn}(\mathrm{Mod}(\Sigma)))(P \Rightarrow (t \approx t'))$. For brevity,

1) For readers familiar with [5]: we consider only the interpretation of Horn clauses which is determined by the globalization [17].

we usually denote $(\mathrm{Horn}(\mathcal{K}))(P \Rightarrow (t \approx t'))$ by $\|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$; if $\mathcal{K} = \{\boldsymbol{M}\}$, $(\mathrm{Horn}(\mathcal{K}))(P \Rightarrow (t \approx t'))$ is denoted by $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}}$. By definition,

$$\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}} = \bigwedge\{\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},v} \mid v : X \longrightarrow M\},$$

i. e. $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}}$ is a degree to which $P \Rightarrow (t \approx t')$ is true in $\boldsymbol{M}$ (under any valuation v).

In [5, 6] we showed that the degree $\|\cdot\|_{\Sigma}$ of semantic consequence can be characterized syntactically. FHL is developed in Pavelka-style, i. e. we define a degree of provability $|\cdot|_{\Sigma}$ and the completeness theorem says that degree of semantic consequence is equal to the degree of provability, i. e. $\|\cdot\|_{\Sigma} = |\cdot|_{\Sigma}$. In addition to that, we have also shown that Horn classes are exactly the classes of $\boldsymbol{L}$-algebras closed under certain class operators, generalizing thus the well-known quasivariety theorem. So far, both the results were limited to classes of structures of truth degrees which do not include the residuated lattices on the unit interval given by left-continuous t-norms.

4 Metric algebras

In [20], N. Weaver studied several topics related to metric algebras. A metric algebra is basically a classical algebra endowed with a metric defined on its universe set. As the author himself pointed out, the key notion of [20] is that of an equicontinuous satisfaction. Weaver showed that classes of metric algebras which equicontinuously satisfy a set of formulas of certain form (implications between so-called atomic inequalities) are exactly the classes of metric algebras closed under subalgebras, isomorphic images, and reduced products. On the other hand, Weaver did not introduce any logical calculus for reasoning with formulas of that form. From the viewpoint of constrained functions, metric algebras which equicontinuously satisfy the compatibility axiom (formulated in terms of metric algebras) can be seen as algebras with functions mapping pairwise close elements to close results. In this section we present a survey of basic notions of metric algebras and equicontinuous satisfaction, and we give some remarks.

Recall that a *metric* ϱ^M *on a set* M is a mapping $\varrho^M : M \times M \longrightarrow [0, \infty]$ satisfying

(i) $\varrho^M(\mathfrak{a}, \mathfrak{b}) = 0$ iff $\mathfrak{a} = \mathfrak{b}$,

(ii) $\varrho^M(\mathfrak{a}, \mathfrak{b}) = \varrho^M(\mathfrak{b}, \mathfrak{a})$,

(iii) $\varrho^M(\mathfrak{a}, \mathfrak{c}) \leq \varrho^M(\mathfrak{a}, \mathfrak{b}) + \varrho^M(\mathfrak{b}, \mathfrak{c})$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in M$. A couple $\langle M, \varrho^M\rangle$, where $M \neq \emptyset$ and ϱ^M is a metric on M, will be called a *metric space*. Note that the notion of a metric is used in the generalized sense, i. e. we allow $\varrho(\cdot,\cdot) = \infty$. A *metric algebra of type* F is a triplet $\boldsymbol{M} = \langle M, \varrho^{\boldsymbol{M}}, F^{\boldsymbol{M}}\rangle$, where

(i) $\langle M, F^{\boldsymbol{M}}\rangle$ is a classical algebra of type F,

(ii) $\langle M, \varrho^{\boldsymbol{M}}\rangle$ is a metric space.

The notion of a metric algebra itself does not include any constraint on functions. In order to introduce the constraint we need to take into account implications between so-called atomic inequalities and the notion of an equicontinuous satisfaction.

An *atomic inequality* is an expression of the form $\varrho(t, t') \preccurlyeq \alpha$, where $t, t' \in T(X)$ and $\alpha \in [0, \infty]$. The atomic inequality $\varrho(t, t') \preccurlyeq \alpha$ is δ*-true in* $\boldsymbol{M}$ *under* v if $\varrho^{\boldsymbol{M}}(\|t\|_{\boldsymbol{M},v}, \|t'\|_{\boldsymbol{M},v}) \leq \alpha+\delta$. $\varrho(t, t') \preccurlyeq \alpha$ is *true in* $\boldsymbol{M}$ *under* v if it is δ-true in $\boldsymbol{M}$ under v for $\delta = 0$. An *implication* (*between atomic inequalities*) is an expression of the form

$$\varrho(s_1, s_1') \preccurlyeq \alpha_1 \,\&\cdots\&\, \varrho(s_n, s_n') \preccurlyeq \alpha_n \Rightarrow \varrho(t, t') \preccurlyeq \beta. \tag{4}$$

Let $\mathcal{K}$ be a class of metric algebras. $\mathcal{K}$ *satisfies* (4) if for each $\boldsymbol{M} \in \mathcal{K}$ and a valuation v we have: if $\varrho(s_i, s_i') \preccurlyeq \alpha_i$ is true in $\boldsymbol{M}$ under v for each $i = 1, \ldots, n$, then $\varrho(t, t') \preccurlyeq \beta$ is true in $\boldsymbol{M}$ under v. $\mathcal{K}$ *satisfies* (4) *equicontinuously* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $\boldsymbol{M} \in \mathcal{K}$ and a valuation v we have: if $\varrho(s_i, s_i') \preccurlyeq \alpha_i$ is δ-true in $\boldsymbol{M}$ under v for each $i = 1, \ldots, n$, then $\varrho(t, t') \preccurlyeq \beta$ is ε-true in $\boldsymbol{M}$ under v.

$\mathcal{K}$ *has equicontinuous functions* if for any n-ary $f \in F$, $\mathcal{K}$ satisfies

$$\varrho(x_1, y_1) \preccurlyeq 0 \,\&\cdots\&\, \varrho(x_n, y_n) \preccurlyeq 0 \Rightarrow \varrho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \preccurlyeq 0 \tag{5}$$

equicontinuously. $\boldsymbol{M}$ *has equicontinuous functions* if $\mathcal{K} = \{\boldsymbol{M}\}$ has equicontinuous functions.

Remark 4.1

(i) If $\mathcal{K}$ satisfies (4) equicontinuously, then $\mathcal{K}$ satisfies (4). In more detail, if each

$$\varrho(s_i, s'_i) \preccurlyeq \alpha_i$$

is true (i. e. 0-true) in $\boldsymbol{M}$ under v, then it is also δ-true. Therefore, if $\varrho(s_i, s'_i) \preccurlyeq \alpha_i$ $(i = 1, \dots, n)$ are true in $\boldsymbol{M}$ under v, then $\varrho(t, t') \preccurlyeq \beta$ is ε-true for each $\varepsilon > 0$ which immediately gives that $\varrho(t, t') \preccurlyeq \beta$ is true (i. e. 0-true).

(ii) Condition (5) can be seen as the classical compatibility axiom (1) which is formulated in terms of metric algebras, i. e. as an implication between atomic inequalities instead of the classical identities.

(iii) In certain cases, equicontinuity of functions of a metric algebra $\boldsymbol{M}$ is trivial. For example, if

$$\{\varrho^{\boldsymbol{M}}(\mathfrak{a}, \mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in M \text{ and } \mathfrak{a} \neq \mathfrak{b}\} \tag{6}$$

has a lower bound $\gamma > 0$, then $\boldsymbol{M}$ has equicontinuous functions since for each $\varepsilon > 0$ one can put $\delta = \frac{\gamma}{2}$. $\boldsymbol{M}$ then satisfies (5) equicontinuously, because $\varrho(x_i, y_i) \preccurlyeq 0$ is δ-true ($\frac{\gamma}{2}$-true) in $\boldsymbol{M}$ under v iff $v(x_i) = v(y_i)$. Thus, if for each $i = 1, \dots, n$ we have that $\varrho(x_i, y_i) \preccurlyeq 0$ is δ-true in $\boldsymbol{M}$ under v, then

$$\varrho(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \preccurlyeq 0$$

is 0-true (and thus ε-true) in $\boldsymbol{M}$ under v. Note that (6) has a lower bound $\gamma > 0$ if M is finite, or more generally, if the set (6) itself is finite.

(iv) $\boldsymbol{M}$ has equicontinuous functions iff functions of $\boldsymbol{M}$ are uniformly continuous.

(v) Let us note that the existence of $\delta > 0$ for each $\varepsilon > 0$ required in the definition of equicontinuous satisfaction applies to the class $\mathcal{K}$. That is, the fact that each $\boldsymbol{M} \in \mathcal{K}$ is uniformly continuous does not imply in general that $\mathcal{K}$ has equicontinuous functions.

5 Algebras with fuzzy equalities vs. metric algebras

We are going to investigate the relationship between algebras with fuzzy equalities and metric algebras with equicontinuous functions. Our motivations are the following. First, in both cases, the constraint on functions is based on a generalization of the classical compatibility (congruence) axiom. Thus, the basic idea behind restricted operations is similar in some sense. Second, for residuated lattices on the unit interval given by continuous Archimedean t-norms, there is a well-known transformation of $\boldsymbol{L}$-similarity spaces to metric spaces and vice versa. It is then natural to ask whether this transformation preserves the compatibility. Before we begin, let us recall the representation of continuous Archimedean t-norms by continuous additive generators.

Theorem 5.1 (see [2, Theorem 7.53]) *A mapping $\otimes : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous Archimedean t-norm iff there is a continuous additive generator g such that*

$$a \otimes b = g^{(-1)}(g(a) + g(b)),$$

i. e. g is a strictly decreasing continuous mapping $g : [0,1] \longrightarrow [0,\infty]$ with $g(1) = 0$ and $g^{(-1)} : [0,\infty] \longrightarrow [0,1]$ is the pseudo-inverse of g defined by

$$g^{(-1)}(a) = \begin{cases} g^{-1}(a) & \text{if } a \leq g(0), \\ 0 & \text{otherwise.} \end{cases}$$

For instance, if $\otimes$ is the Goguen (product) conjunction, then $g(a) = -\log(a)$ and $g^{(-1)}(a) = \exp(-a)$; if $\otimes$ is the Łukasiewicz conjunction, then $g(a) = 1 - a$ and $g^{(-1)}(a) = \max(1 - a, 0)$, etc. The following theorem describes the well-known relationship between similarity spaces and metric spaces.

Theorem 5.2 (see [2, Theorem 7.54]) *Let $\otimes$ be a continuous Archimedean t-norm with the additive generator g, $\boldsymbol{L}$ be the complete residuated lattice on $[0,1]$ given by $\otimes$, $\approx^M$ be an $\boldsymbol{L}$-equality on M, ϱ^M be a metric on M. For each $\mathfrak{a}, \mathfrak{b} \in M$ put*

$$\varrho_{\approx^M}(\mathfrak{a}, \mathfrak{b}) = g(\mathfrak{a} \approx^M \mathfrak{b}), \quad \mathfrak{a} \approx_{\varrho^M} \mathfrak{b} = g^{(-1)}(\varrho^M(\mathfrak{a}, \mathfrak{b})). \tag{7}$$

Then $\varrho_{\approx^M} : [0,1] \times [0,1] \longrightarrow [0,\infty]$ is a metric on M and $\approx_{\varrho^M} : [0,1] \times [0,1] \longrightarrow [0,1]$ is an $\boldsymbol{L}$-equality on M. Furthermore, $\approx^M$ equals $\approx_{\varrho_{\approx^M}}$ and if $\{\varrho^M(\mathfrak{a}, \mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in M\} \subseteq [0, g(0)]$, then ϱ^M equals $\varrho_{\approx_{\varrho^M}}$.

We are going to show that $\boldsymbol{L}$-algebras with $\boldsymbol{L}$ given by a continuous Archimedean t-norm can be turned into metric algebras with equicontinuous functions.

Lemma 5.3 *Let $\boldsymbol{L}$ be the residuated lattice given by a continuous Archimedean t-norm $\otimes$, $\langle M, \approx^M\rangle$ be an $\boldsymbol{L}$-similarity space, $f^M : M^n \longrightarrow M$ be a mapping compatible with $\approx^M$. Then*

$$\varrho_{\approx^M}(f^M(\mathfrak{a}_1,\dots,\mathfrak{a}_n), f^M(\mathfrak{b}_1,\dots,\mathfrak{b}_n)) \le \varrho_{\approx^M}(\mathfrak{a}_1,\mathfrak{b}_1) + \cdots + \varrho_{\approx^M}(\mathfrak{a}_n,\mathfrak{b}_n)$$

for each $\mathfrak{a}_1,\mathfrak{b}_1,\dots,\mathfrak{a}_n,\mathfrak{b}_n \in M$.

P r o o f. Let g be the additive generator of $\otimes$. First, we claim that for each $a_1,\dots,a_n \in L$ we have

$$g(a_1 \otimes \cdots \otimes a_n) \le g(a_1) + \cdots + g(a_n). \tag{8}$$

For $n = 1$, the claim is trivial. By induction, suppose $g(a_2 \otimes \cdots \otimes a_n) \le g(a_2) + \cdots + g(a_n)$. We have

$$g(\textstyle\bigotimes_{i=1}^n a_i) = g(g^{(-1)}(g(a_1) + g(\bigotimes_{i=2}^n a_i))).$$

Now if $g(a_1) + g(\bigotimes_{i=2}^n a_i) \le g(0)$, we have

$$g(\textstyle\bigotimes_{i=1}^n a_i) = g(g^{-1}(g(a_1) + g(\bigotimes_{i=2}^n a_i))) = g(a_1) + g(\bigotimes_{i=2}^n a_i) \le \sum_{i=1}^n g(a_i).$$

If $g(a_1) + g(\bigotimes_{i=2}^n a_i) \not\le g(0)$, we have $g(0) < g(a_1) + g(\bigotimes_{i=2}^n a_i)$ and $g^{(-1)}(g(a_1) + g(\bigotimes_{i=2}^n a_i)) = 0$, i.e.

$$g(\textstyle\bigotimes_{i=1}^n a_i) = g(0) < g(a_1) + g(\bigotimes_{i=2}^n a_i) \le \sum_{i=1}^n g(a_i),$$

showing that (8) holds. Thus, applying this claim and the fact that g is decreasing, one can get

$$\begin{aligned}\varrho_{\approx^M}(f^M(\mathfrak{a}_1,\dots,\mathfrak{a}_n), f^M(\mathfrak{b}_1,\dots,\mathfrak{b}_n)) &= g(f^M(\mathfrak{a}_1,\dots,\mathfrak{a}_n) \approx^M f^M(\mathfrak{b}_1,\dots,\mathfrak{b}_n))\\ &\le g(\mathfrak{a}_1 \approx^M \mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{a}_n \approx^M \mathfrak{b}_n)\\ &\le g(\mathfrak{a}_1 \approx^M \mathfrak{b}_1) + \cdots + g(\mathfrak{a}_n \approx^M \mathfrak{b}_n)\\ &= \varrho_{\approx^M}(\mathfrak{a}_1,\mathfrak{b}_1) + \cdots + \varrho_{\approx^M}(\mathfrak{a}_n,\mathfrak{b}_n),\end{aligned}$$

which is the desired inequality. □

Theorem 5.4 *Let $\boldsymbol{L}$ be the residuated lattice given by a continuous Archimedean t-norm $\otimes$, $\langle M, \approx^{\boldsymbol{M}}, F^{\boldsymbol{M}}\rangle$ be an $\boldsymbol{L}$-algebra. Then $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ is a metric algebra with equicontinuous functions.*

P r o o f. Theorem 5.2 yields that $\varrho_{\approx^{\boldsymbol{M}}}$ is a metric since $\approx^{\boldsymbol{M}}$ is an $\boldsymbol{L}$-equality. Thus, it remains to check that for each n-ary $f \in F$, $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ satisfies (5) equicontinuously. Recall that an n-ary function symbol f is interpreted by an n-ary function $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$. If f is a nullary function symbol, the claim is trivial. So suppose $n > 0$. For each $\varepsilon > 0$ put $\delta = \frac{\varepsilon}{n}$. Then, assuming $\varrho_{\approx^{\boldsymbol{M}}}(\mathfrak{a}_i,\mathfrak{b}_i) \le \delta$ for each $i = 1,\dots,n$, Lemma 5.3 gives

$$\varrho_{\approx^{\boldsymbol{M}}}(f^{\boldsymbol{M}}(\mathfrak{a}_1,\dots,\mathfrak{a}_n), f^{\boldsymbol{M}}(\mathfrak{b}_1,\dots,\mathfrak{b}_n)) \le \varrho_{\approx^{\boldsymbol{M}}}(\mathfrak{a}_1,\mathfrak{b}_1) + \cdots + \varrho_{\approx^{\boldsymbol{M}}}(\mathfrak{a}_n,\mathfrak{b}_n) \le n\delta = \varepsilon.$$

Therefore, if each $\varrho(x_i, y_i) \preccurlyeq 0$ is δ-true in $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ under v, then

$$\varrho(f(x_1,\dots,x_n), f(y_1,\dots,y_n)) \preccurlyeq 0$$

is ε-true in $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ under v. Hence, $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ satisfies (5) equicontinuously. □

Corollary 5.5 *Let $\boldsymbol{L}$ be the residuated lattice given by a continuous Archimedean t-norm $\otimes$, $\langle M, \approx^{\boldsymbol{M}}, F^{\boldsymbol{M}}\rangle$ be an $\boldsymbol{L}$-algebra. Then each $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$ is uniformly continuous with respect to $\varrho_{\approx^{\boldsymbol{M}}}$.*

To sum up, we have shown that each $\boldsymbol{L}$-algebra with $\boldsymbol{L}$ given by a continuous Archimedean t-norm $\otimes$ can be transformed into a metric algebra with continuous functions using the additive generator g of $\otimes$. The converse transformation is not possible in general. Namely, one cannot use the pseudo-inverse $g^{(-1)}$ of g to transform a metric algebra with equicontinuous functions into an $\boldsymbol{L}$-algebra. An example follows.

Example 5.6 For each complete residuated lattice $\boldsymbol{L}$ given by a continuous Archimedean t-norm, one can find a finite metric algebra $\boldsymbol{M} = \langle M, \varrho^{\boldsymbol{M}}, F^{\boldsymbol{M}}\rangle$ with a single unary function $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$ such that $f^{\boldsymbol{M}}$ is not compatible with $\approx_{\varrho^{\boldsymbol{M}}}$. For instance, put $M = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, and let $f^{\boldsymbol{M}}$ and $\approx^{\boldsymbol{M}}$ be defined by the following tables:

	$f^{\boldsymbol{M}}$
$\mathfrak{a}$	$\mathfrak{c}$
$\mathfrak{b}$	$\mathfrak{d}$
$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{d}$	$\mathfrak{d}$

$\approx^{\boldsymbol{M}}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$
$\mathfrak{a}$	1	x	0	0
$\mathfrak{b}$	x	1	0	0
$\mathfrak{c}$	0	0	1	y
$\mathfrak{d}$	0	0	y	1 .

Suppose $1 > x > y$. It is easily seen that $\approx^{\boldsymbol{M}}$ is an $\boldsymbol{L}$-equality and $\approx^{\boldsymbol{M}}$ equals $\approx_{\varrho_{\approx^{\boldsymbol{M}}}}$ by Theorem 5.2. Since M is finite, we have that $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ is a metric algebra with equicontinuous functions, see Remark 4.1(iii). On the other hand, $f^{\boldsymbol{M}}$ is not compatible with $\approx_{\varrho_{\approx^{\boldsymbol{M}}}}$ (i. e. with $\approx^{\boldsymbol{M}}$), because

$$\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} = x \nleq y = \mathfrak{c} \approx^{\boldsymbol{M}} \mathfrak{d} = f(\mathfrak{a}) \approx^{\boldsymbol{M}} f(\mathfrak{b}).$$

Therefore, $\langle M, \varrho_{\approx^{\boldsymbol{M}}}, F^{\boldsymbol{M}}\rangle$ cannot be turned into an $\boldsymbol{L}$-algebra by using (7).

The present section showed that even if the idea behind equicontinuity of operations in metric algebras is similar to that of compatibility with a given $\boldsymbol{L}$-equality, the theory of metric algebras deals with a completely different type of restrictions on operations. On the one hand, $\boldsymbol{L}$-algebras can be thought of as more general than metric algebras because $\boldsymbol{L}$ need not be linearly ordered. On the other hand, if we restrict ourselves only to $\boldsymbol{L}$'s given by continuous Archimedean t-norms, compatibility with a given $\boldsymbol{L}$-equality is more restrictive than the equicontinuity of functions as we have seen in Theorem 5.4 and Example 5.6.

6 Continuous Horn classes

In this section we first define the notion of equicontinuity of functions in the context of algebras with fuzzy equalities without invoking a connection to metric algebras. The motivation for doing so is basically the following. The translation of algebras with fuzzy equalities into metric algebras introduced in the previous section is established only for continuous Archimedean t-norms. Furthermore, the converse translation is not possible in general. Since we are interested in characterizing Horn classes (by their closure properties) as general as possible, we wish to use any left-continuous t-norm. In the sequel, we are going to establish a characterization of quasivarieties alternative to that presented in [6]. The characterization presented below uses the notion of continuity together with reduced products [19] and covers all residuated lattices given by left-continuous t-norms. The notion of continuity itself will be defined using an arbitrary (but fixed) strict continuous Archimedean t-norm.

Let us briefly recall some basic structural notions of $\boldsymbol{L}$-algebras which have already been introduced before, see [2, 3, 4, 19]. An $\boldsymbol{L}$-algebra $\boldsymbol{N} = \langle N, \approx^{\boldsymbol{N}}, F^{\boldsymbol{N}}\rangle$ is called a *subalgebra of an $\boldsymbol{L}$-algebra $\boldsymbol{M}$* if $\langle N, F^{\boldsymbol{N}}\rangle$ is a subalgebra of $\langle M, F^{\boldsymbol{M}}\rangle$ (as a classical algebra) and $\approx^{\boldsymbol{N}}$ is a restriction of $\approx^{\boldsymbol{M}}$ to N. Let $\boldsymbol{M}, \boldsymbol{N}$ be $\boldsymbol{L}$-algebras of the same type. A mapping $h : M \longrightarrow N$ satisfying $\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} \leq h(\mathfrak{a}) \approx^{\boldsymbol{N}} h(\mathfrak{b})$ is called an *$\approx$-morphism.* An $\approx$-morphism $h : M \longrightarrow N$ is called a *morphism* (*of $\boldsymbol{L}$-algebras*) if h, denoted $h : \boldsymbol{M} \longrightarrow \boldsymbol{N}$, is a morphism between classical algebras $\langle M, F^{\boldsymbol{M}}\rangle$ and $\langle N, F^{\boldsymbol{N}}\rangle$. A morphism $h : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ satisfying (for $\mathfrak{a}, \mathfrak{b} \in M$)

$$\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} = h(\mathfrak{a}) \approx^{\boldsymbol{N}} h(\mathfrak{b})$$

is called an *embedding*. A surjective embedding is called an *isomorphism*. For an isomorphism $h : \boldsymbol{M} \longrightarrow \boldsymbol{N}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ are called *isomorphic*; $\boldsymbol{N}$ is an *isomorphic image* of $\boldsymbol{M}$. An $\boldsymbol{L}$-relation θ on M such that

(i) θ is an $\boldsymbol{L}$-equivalence on M,

(ii) $\approx^{\boldsymbol{M}} \subseteq \theta$,

(iii) all functions $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$ are compatible with θ,

is called a *congruence on $\boldsymbol{M}$*. For a congruence θ on an $\boldsymbol{L}$-algebra $\boldsymbol{M}$, $\boldsymbol{L}$-algebra

$$\boldsymbol{M}/\theta = \langle M/\theta, \approx^{\boldsymbol{M}/\theta}, F^{\boldsymbol{M}/\theta}\rangle,$$

where

(i) $\langle M/\theta, F^{\boldsymbol{M}/\theta}\rangle$ is the classical factor algebra of $\langle M, F^{\boldsymbol{M}}\rangle$ modulo $\{\langle \mathfrak{a}, \mathfrak{b}\rangle \mid \theta(\mathfrak{a}, \mathfrak{b}) = 1\}$,

(ii) $[\mathfrak{a}]_\theta \approx^{\boldsymbol{M}/\theta} [\mathfrak{b}]_\theta = \theta(\mathfrak{a}, \mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in M$,

is called *the factor $\boldsymbol{L}$-algebra of $\boldsymbol{M}$ modulo θ. The direct product* $\prod_{i\in I} \boldsymbol{M}_i = \langle \prod_{i\in I} M_i, \approx^{\prod_{i\in I} \boldsymbol{M}_i}, F^{\prod_{i\in I} \boldsymbol{M}_i}\rangle$ *of $\boldsymbol{L}$-algebras $\boldsymbol{M}_i$ ($i \in I$)* is an $\boldsymbol{L}$-algebra such that $\langle \prod_{i\in I} M_i, F^{\prod_{i\in I} \boldsymbol{M}_i}\rangle$ is the direct product of classical algebras $\langle M_i, F^{\boldsymbol{M}_i}\rangle$ and $\approx^{\prod_{i\in I} \boldsymbol{M}_i}$ is defined by $\mathfrak{a} \approx^{\prod_{i\in I} \boldsymbol{M}_i} \mathfrak{b} = \bigwedge_{i\in I} \mathfrak{a}(i) \approx^{\boldsymbol{M}_i} \mathfrak{b}(i)$. Let $\{\boldsymbol{M}_i \mid i \in I\}$ be a family of $\boldsymbol{L}$-algebras and let F be a filter over I. We define a binary $\boldsymbol{L}$-relation θ_F on $\prod_{i\in I} M_i$ by

$$\theta_F(\mathfrak{a}, \mathfrak{b}) = \bigvee\nolimits_{X\in F} \bigwedge\nolimits_{i\in X} \mathfrak{a}(i) \approx^{\boldsymbol{M}_i} \mathfrak{b}(i).$$

One can show that θ_F is a congruence [19]. The factor $\boldsymbol{L}$-algebra $(\prod_{i\in I} \boldsymbol{M}_i)/\theta_F$, denoted by $\prod_F \boldsymbol{M}_i$, is called *the reduced product of $\{\boldsymbol{M}_i \mid i \in I\}$ modulo F*. Note that in particular if $I = \emptyset$, then $\prod_F \boldsymbol{M}_i$ is a trivial (one-element) $\boldsymbol{L}$-algebra.

In what follows, $\boldsymbol{L}$ denotes a residuated lattice on $[0, 1]$ given by a left-continuous t-norm $\otimes$.

Definition 6.1 Let $\odot$ be a strict continuous Archimedean t-norm. For a finite binary $\boldsymbol{L}$-relation P on $T(X)$ and $d \in [0, 1]$ we define a finite binary $\boldsymbol{L}$-relation $d \odot P$ on $T(X)$ by $(d \odot P)(r, r') = d \odot P(r, r')$ for all $r, r' \in T(X)$.

Let $\mathcal{K}$ be a class of $\boldsymbol{L}$-algebras and let $P \Rightarrow (t \approx t')$ be a Horn clause. $\mathcal{K}$ *satisfies $P \Rightarrow (t \approx t')$ $\odot$-continuously* if for each $e \in [0, 1)$ there is $d \in [0, 1)$ such that

$$e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \leq \|d \odot P \Rightarrow (t \approx t')\|_{\mathcal{K}}.$$

$\mathcal{K}$ is *$\odot$-continuous* if $\mathcal{K}$ satisfies all Horn clauses $\odot$-continuously.

Remark 6.2 Note that in general, $\odot$ has nothing to do with $\otimes$. We just use two conjunctions simultaneously: the left-continuous t-norm $\otimes$ determines the structure of truth degrees $\boldsymbol{L}$ while the strict continuous t-norm $\odot$ determines the notion of $\odot$-continuity.

The following technical lemma describes properties of $\odot$ that will be used further.

Lemma 6.3 *Let $x, y \in [0, 1]$.*

(i) *If $x \geq y \odot z$ for each $z \in [0, 1)$, then $x \geq y$.*

(ii) *If $x < y$, then there is $z \in [0, 1)$ such that $x < y \odot z$.*

Proof.

(i) If $x \geq y \odot z$ for each $z \neq 1$, then $x \geq \bigvee\{y \odot z \mid z \neq 1\} = y \odot \bigvee\{z \mid z \neq 1\} = y \odot 1 = y$.

(ii) is a restatement of (i). □

Theorem 6.4 *If $\mathcal{K}$ is a $\odot$-continuous Horn class, then $\mathcal{K}$ is closed under isomorphic images, subalgebras, and reduced products.*

Proof. Let $\mathcal{K}$ be a $\odot$-continuous class of $\boldsymbol{L}$-algebras such that $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some $\boldsymbol{L}$-set Σ of Horn clauses. Obviously, $\mathcal{K}$ is closed under isomorphic images because if $\boldsymbol{M}$ and $\boldsymbol{N}$ are isomorphic, then, for each $P \Rightarrow (t \approx t')$, $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}} = \|P \Rightarrow (t \approx t')\|_{\boldsymbol{N}}$. $\mathcal{K}$ is closed under subalgebras because if $\boldsymbol{N}$ is a subalgebra of $\boldsymbol{M}$, then, for each $P \Rightarrow (t \approx t')$, $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}} \leq \|P \Rightarrow (t \approx t')\|_{\boldsymbol{N}}$, see [6] for details. So it remains to check that $\mathcal{K}$ is closed under reduced products.

Take a family $\{\boldsymbol{M}_i \in \mathcal{K} \mid i \in I\}$ of $\boldsymbol{L}$-algebras and a filter F over I. It is sufficient to check that

$$\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i} \geq \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \tag{9}$$

is true for each $P \Rightarrow (t \approx t')$, because $\mathcal{K} = \mathrm{Mod}(\Sigma)$ yields $\|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \geq \Sigma(P \Rightarrow (t \approx t'))$ which together with (9) gives

$$\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i} \geq \Sigma(P \Rightarrow (t \approx t')),$$

i. e. $\prod_F \boldsymbol{M}_i \in \mathrm{Mod}(\Sigma) = \mathcal{K}$.

We proceed by checking that

$$\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i, w} \geq \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \tag{10}$$

is true for all valuations $w : X \longrightarrow (\prod_{i \in I} M_i)/\theta_F$ (recall that $(\prod_{i \in I} M_i)/\theta_F$ is a support of $\prod_F \boldsymbol{M}_i$). Following the definition of $\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i, w}$, we distinguish two situations. First, if there are $s, s' \in T(X)$ such that $P(s, s') \nleq \|s \approx s'\|_{\prod_F \boldsymbol{M}_i, w}$, then $\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i, w} = 1$ and (10) is trivially satisfied. Second, if $P(s, s') \leq \|s \approx s'\|_{\prod_F \boldsymbol{M}_i, w}$ for all terms $s, s' \in T(X)$, then $\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_i, w} = \|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w}$ and we have to check that $\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} \geq \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$. Thus, it is enough to assume

$$\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} < \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$$

and show that there are terms $s, s' \in T(X)$ such that $P(s, s') \nleq \|s \approx s'\|_{\prod_F \boldsymbol{M}_i, w}$.

So, let $\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} < \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$. Using Lemma 6.3(ii), there is $e \in [0, 1)$ such that

$$\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} < e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}. \tag{11}$$

Since $\mathcal{K}$ is $\odot$-continuous, there is $d \in [0, 1)$ such that

$$\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} < e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \leq \|d \odot P \Rightarrow (t \approx t')\|_{\mathcal{K}}.$$

We now prove the following claim: there are $s, s' \in T(X)$ with $P(s, s') \neq 0$ such that for each $Z \in F$ we have

$$\textstyle\bigwedge_{i \in Z} \|s \approx s'\|_{\boldsymbol{M}_i, v_i} < (d \odot P)(s, s'), \tag{12}$$

where $v_i : X \longrightarrow M_i$ is a valuation defined by $v_i(x) = (v(x))(i)$, and $v : X \longrightarrow \prod_{i \in I} M_i$ is a mapping with $v(x) \in w(x)$ for all $x \in X$. This claim can be proven by contradiction: assume that for all $s, s' \in T(X)$ with $P(s, s') \neq 0$ there is $Z_{s,s'} \in F$ such that $\bigwedge_{i \in Z_{s,s'}} \|s \approx s'\|_{\boldsymbol{M}_i, v_i} \geq (d \odot P)(s, s')$. Since P is finite and F is a filter, for $Z' = \bigcap_{P(s,s') \neq 0} Z_{s,s'}$ we have $Z' \in F$. So, for all $s, s' \in T(X)$ such that $P(s, s') \neq 0$, we have

$$\textstyle\bigwedge_{i \in Z'} \|s \approx s'\|_{\boldsymbol{M}_i, v_i} \geq \bigwedge_{i \in Z_{s,s'}} \|s \approx s'\|_{\boldsymbol{M}_i, v_i} \geq (d \odot P)(s, s').$$

If $P(s, s') = 0$, then trivially $(d \odot P)(s, s') \leq \|s \approx s'\|_{\boldsymbol{M}_i, v_i}$. Thus, $(d \odot P)(s, s') \leq \|s \approx s'\|_{\boldsymbol{M}_i, v_i}$ is true for all $s, s' \in T(X)$, and any $i \in Z'$. As a further consequence, $\|d \odot P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_i, v_i} = \|t \approx t'\|_{\boldsymbol{M}_i, v_i}$ $(i \in Z')$. Since $\mathcal{K}$ is $\odot$-continuous, we have

$$\begin{aligned}
\textstyle\bigwedge_{i \in Z'} \|t \approx t'\|_{\boldsymbol{M}_i, v_i} &= \textstyle\bigwedge_{i \in Z'} \|d \odot P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_i, v_i} \\
&\geq \|d \odot P \Rightarrow (t \approx t')\|_{\mathcal{K}} \\
&\geq e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}.
\end{aligned}$$

This immediately gives

$$\|t \approx t'\|_{\prod_F \boldsymbol{M}_i, w} = \textstyle\bigvee_{Z \in F} \bigwedge_{i \in Z} \|t \approx t'\|_{\boldsymbol{M}_i, v_i} \geq \bigwedge_{i \in Z'} \|t \approx t'\|_{\boldsymbol{M}_i, v_i} \geq e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$$

which violates (11). Hence, claim (12) is true. Using claim (12), there are $s, s' \in T(X)$ for which

$$\|s \approx s'\|_{\prod_F \boldsymbol{M}_i, w} = \textstyle\bigvee_{Z \in F} \bigwedge_{i \in Z} \|s \approx s'\|_{\boldsymbol{M}_i, v_i} \leq (d \odot P)(s, s') < P(s, s')$$

showing that $P(s, s') \nleq \|s \approx s'\|_{\prod_F \boldsymbol{M}_i, w}$ which finishes the proof. □

So far, we used a single set of variables X which was supposed to be denumerable. Now, for technical reasons, we need to consider Horn clauses constituted of identities in various sets of variables. If $P \Rightarrow (t \approx t')$ is a Horn clause of the form (2) and if $s_1, s_1', \ldots, s_n, s_n', t, t' \in T(Y)$, then $P \Rightarrow (t \approx t')$ will be called a *Horn clause in variables* Y. $\mathrm{Horn}_Y(\mathcal{K})$ will denote an $\boldsymbol{L}$-set of Horn clauses in variables Y defined by (3).

In what follows, we are going to use support sets of $\boldsymbol{L}$-algebras as sets of variables. For instance, if $\boldsymbol{M}$ is an $\boldsymbol{L}$-algebra with support set M, we consider M as a set of variables. The identical mapping $\mathrm{id}_M : M \longrightarrow M$ then becomes a particular valuation of variables from M. For convenience, we denote the elements $\mathfrak{a}, \mathfrak{b}, \ldots$ of M also by $a, b, \ldots$.

Lemma 6.5 *Let X be a denumerable set of variables, $\mathcal{K}$ be a class of $\boldsymbol{L}$-algebras, $\boldsymbol{M} \in \mathrm{Mod}(\mathrm{Horn}_X(\mathcal{K}))$, $P \Rightarrow (t \approx t')$ be a Horn clause in variables M. If $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},\mathrm{id}_M} < c$, then there is $\boldsymbol{N} \in \mathcal{K}$ and a valuation $v : M \longrightarrow N$ such that $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{N},v} < c$.*

Proof. Let $P \Rightarrow (t \approx t')$ be a Horn clause of the form (2), where $s_1, s'_1, \ldots, s_n, s'_n, t, t' \in T(M)$. Denote the set of all variables occurring in $s_1, s'_1, \ldots, s_n, s'_n, t, t'$ by M'. Since M' is a finite subset of M and X is a denumerable set of variables, we can fix an injective mapping $g : M' \longrightarrow T(X)$ such that, for each $m \in M'$, $g(m) \in X$. Moreover, we can consider the (uniquely given) homomorphic extension $h : T(M') \longrightarrow T(X)$ of g, and the following Horn clause corresponding to $P \Rightarrow (t \approx t')$:

$$(13) \qquad \langle h(s_1) \approx h(s'_1), a_1\rangle \,\&\cdots\&\langle h(s_n) \approx h(s'_n), a_n\rangle \Rightarrow (h(t) \approx h(t')).$$

Formula (13), abbreviated by $h(P) \Rightarrow (h(t) \approx h(t'))$, is a Horn clause in variables X which, loosely speaking, results from $P \Rightarrow (t \approx t')$ by a consistent renaming of its variables (each variable $m \in M'$ which occurs in $P \Rightarrow (t \approx t')$ is replaced by a variable $g(m) \in X$). Take some $\overline{m} \in M$ and define a valuation $w : X \longrightarrow M$ by

$$w(x) = \begin{cases} m & \text{if } m \in M' \text{ and } g(m) = x, \\ \overline{m} & \text{otherwise.} \end{cases}$$

For each term $r \in T(M')$, $\|r\|_{\boldsymbol{M},\mathrm{id}_M} = \|h(r)\|_{\boldsymbol{M},w}$, i. e. we get

$$\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},\mathrm{id}_M} = \|h(P) \Rightarrow (h(t) \approx h(t'))\|_{\boldsymbol{M},w}.$$

By the latter equality, if $\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},\mathrm{id}_M} < c$, then

$$\|h(P) \Rightarrow (h(t) \approx h(t'))\|_{\boldsymbol{M},w} < c.$$

Since $\boldsymbol{M} \in \mathrm{Mod}(\mathrm{Horn}_X(\mathcal{K}))$ and $h(P) \Rightarrow (h(t) \approx h(t'))$ is a Horn clause in variables X, there must be an $\boldsymbol{L}$-algebra $\boldsymbol{N} \in \mathcal{K}$ and a valuation $w' : X \longrightarrow N$ such that $\|h(P) \Rightarrow (h(t) \approx h(t'))\|_{\boldsymbol{N},w'} < c$. Take some $\overline{n} \in N$ and define a valuation $v : M \longrightarrow N$ by

$$v(m) = \begin{cases} w'(g(m)) & \text{if } m \in M', \\ \overline{n} & \text{otherwise.} \end{cases}$$

For each term $r \in T(M')$, $\|r\|_{\boldsymbol{N},v} = \|h(r)\|_{\boldsymbol{N},w'}$. As a consequence,

$$\|P \Rightarrow (t \approx t')\|_{\boldsymbol{N},v} = \|h(P) \Rightarrow (h(t) \approx h(t'))\|_{\boldsymbol{N},w'} < c,$$

which is the desired (strict) inequality. □

In the sequel, we need the following notion. A pair $\langle P \Rightarrow (t \approx t'), a\rangle$, where $P \Rightarrow (t \approx t')$ is a Horn clause (in variables X) and $a \in L$ (i. e. a is a truth-degree) will be called a *truth-weighted Horn clause* (*in variables X*).

Theorem 6.6 *If $\mathcal{K}$ is closed under isomorphic images, subalgebras, and reduced products, then $\mathcal{K}$ is a Horn class.*

Proof. Let X be a denumerable set of variables. Put $\Sigma = \mathrm{Horn}_X(\mathcal{K})$. Described verbally, Σ is an $\boldsymbol{L}$-set of Horn clauses in variables X such that $\Sigma(P \Rightarrow (t \approx t'))$ is a degree to which $P \Rightarrow (t \approx t')$ is true in $\mathcal{K}$. We are going to prove that $\mathcal{K} = \mathrm{Mod}(\Sigma)$. Obviously, $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma)$. Thus, we check the converse inclusion.

Take $\boldsymbol{M} \in \mathrm{Mod}(\Sigma)$. If $\boldsymbol{M}$ is trivial (i. e. one-element $\boldsymbol{L}$-algebra), then $\boldsymbol{M} \in \mathcal{K}$ because $\mathcal{K}$ is closed under reduced products. So, let $\boldsymbol{M}$ be nontrivial and consider M (the universe of $\boldsymbol{M}$) as a set of variables. Introduce an index set I, whose elements are truth-weighted Horn clauses in variables M, as follows:

$$I = \{\langle P \Rightarrow (t \approx t'), c\rangle \mid \|P \Rightarrow (t \approx t')\|_{\boldsymbol{M},\mathrm{id}_M} < c\}.$$

A truth-weighted Horn clause $\langle P \Rightarrow (t \approx t'), c\rangle$ in variables M belongs to I iff the degree to which $P \Rightarrow (t \approx t')$ is true in $\boldsymbol{M}$ under id_M is strictly less than c. The nontriviality of $\boldsymbol{M}$ yields $I \neq \emptyset$ because, e. g.,

$$\langle \emptyset \Rightarrow (a \approx b), 1\rangle \in I$$

for any distinct $\mathfrak{a}, \mathfrak{b} \in M$. Since $\boldsymbol{M} \in \mathrm{Mod}(\Sigma) = \mathrm{Mod}(\mathrm{Horn}_X(\mathcal{K}))$, Lemma 6.5 gives that for each

$$\langle P \Rightarrow (t \approx t'), c\rangle \in I,$$

abbreviated by $\langle \varphi, c\rangle$, there is $\boldsymbol{M}_{\langle \varphi, c\rangle} \in \mathcal{K}$ and a valuation $v_{\langle \varphi, c\rangle} : M \longrightarrow M_{\langle \varphi, c\rangle}$ such that

$$\|P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}} < c.$$

That is, for all $s, s' \in T(M)$, we have

(14) $$P(s, s') \leq \|s \approx s'\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}} \quad \text{and} \quad \|t \approx t'\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}} < c.$$

For $r, r' \in T(M)$, consider a subset $Z_{r,r'} \subseteq I$ defined by

(15) $$Z_{r,r'} = \{\langle P \Rightarrow (t \approx t'), c\rangle \in I \mid P(r, r') = \|r \approx r'\|_{\boldsymbol{M}, \mathrm{id}_M}\}.$$

Clearly, $Z_{r,r'} \neq \emptyset$. In addition to that, $\{Z_{r,r'} \mid r, r' \in T(M)\}$ has the finite intersection property [8]. Now, let F be the proper filter over I generated by $\{Z_{r,r'} \mid r, r' \in T(M)\}$. Furthermore, let $\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}$ be the reduced product of $\{\boldsymbol{M}_{\langle \varphi, c\rangle} \mid \langle \varphi, c\rangle \in I\}$ modulo F and let $w : M \longrightarrow (\prod_{\langle \varphi, c\rangle \in I} M_{\langle \varphi, c\rangle})/\theta_F$ be the valuation induced by $v_{\langle \varphi, c\rangle}$'s. That is, $w(x) = [v(x)]_{\theta_F}$, where $v : M \longrightarrow \prod_{\langle \varphi, c\rangle \in I} M_{\langle \varphi, c\rangle}$ such that $(v(x))(\langle \varphi, c\rangle) = v_{\langle \varphi, c\rangle}(x)$ for all $x \in M$, and $\langle \varphi, c\rangle \in I$. The key point here is to show that w is an embedding.

It is routine to check that w is compatible with functions: for any n-ary $f^{\boldsymbol{M}}$, and elements $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in M$ we have $w(f^{\boldsymbol{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)) = f^{\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}}(w(\mathfrak{a}_1), l \ldots, w(\mathfrak{a}_n))$. We now show that w is an $\approx$-morphism. For each $\mathfrak{a}, \mathfrak{b} \in M$ consider $Z_{a,b} \in F$, i. e.

$$Z_{a,b} = \{\langle P \Rightarrow (t \approx t'), c\rangle \in I \mid P(a, b) = \mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b}\}$$

is a particular case of (15) because $a, b \in T(M)$. Using (14) we get that $\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} \leq \|a \approx b\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}}$ is true for each $\langle \varphi, c\rangle \in Z_{a,b}$. Thus,

$$\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} \leq \textstyle\bigwedge_{\langle \varphi, c\rangle \in Z_{a,b}} \|a \approx b\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}},$$

yielding

$$\begin{aligned} \mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} &\leq \textstyle\bigvee_{Z \in F} \bigwedge_{\langle \varphi, c\rangle \in Z} \|a \approx b\|_{\boldsymbol{M}_{\langle \varphi, c\rangle}, v_{\langle \varphi, c\rangle}} \\ &= \|a \approx b\|_{\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}, w} \\ &= w(\mathfrak{a}) \approx^{\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}} w(\mathfrak{b}). \end{aligned}$$

Hence, w is a morphism from $\boldsymbol{L}$-algebra $\boldsymbol{M}$ to $\boldsymbol{L}$-algebra $\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}$. We continue by showing that

(16) $$\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} = w(\mathfrak{a}) \approx^{\prod_F \boldsymbol{M}_{\langle \varphi, c\rangle}} w(\mathfrak{b}).$$

For $\mathfrak{a} = \mathfrak{b}$, (16) is trivial. Take $\mathfrak{a}, \mathfrak{b} \in M$ such that $\mathfrak{a} \neq \mathfrak{b}$. Consider any $c > \mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b}$. We have

$$\|a \approx b\|_{\boldsymbol{M}, \mathrm{id}_M} < c.$$

Observe that for $Z_{r,r'} \in F$ we have that if $\langle P \Rightarrow (t \approx t'), d\rangle \in Z_{r,r'}$, then $\langle P \Rightarrow (a \approx b), c\rangle \in Z_{r,r'}$. Since F is a filter generated by $\{Z_{r,r'} \mid r, r' \in T(M)\}$, for each $Z \in F$ there is P such that $\langle P \Rightarrow (a \approx b), c\rangle \in Z$. Therefore,

$$\textstyle\bigwedge_{\langle \varphi, d\rangle \in Z} \|a \approx b\|_{\boldsymbol{M}_{\langle \varphi, d\rangle}, v_{\langle \varphi, d\rangle}} < c.$$

Thus, $\bigwedge_{\langle\varphi,d\rangle\in Z}\|a \approx b\|_{\boldsymbol{M}_{\langle\varphi,d\rangle},v_{\langle\varphi,d\rangle}} \leq \mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b}$ for all $Z \in F$ because c was chosen arbitrarily. We now have

$$\begin{aligned} w(\mathfrak{a}) \approx^{\prod_F \boldsymbol{M}_{\langle\varphi,c\rangle}} w(\mathfrak{b}) &= \|a \approx b\|_{\prod_F \boldsymbol{M}_{\langle\varphi,c\rangle},w} \\ &= \bigvee_{Z\in F} \bigwedge_{\langle\varphi,c\rangle\in Z}\|a \approx b\|_{\boldsymbol{M}_{\langle\varphi,c\rangle},v_{\langle\varphi,c\rangle}} \\ &\leq \mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b}. \end{aligned}$$

Altogether, w is an embedding. Thus, $\boldsymbol{M}$ is isomorphic to a subalgebra of $\prod_F \boldsymbol{M}_{\langle\varphi,c\rangle}$, i. e. $\boldsymbol{M} \in \mathcal{K}$ because $\mathcal{K}$ is closed under isomorphic images, subalgebras, and reduced products. □

Theorem 6.7 *If $\mathcal{K}$ is closed under reduced products, then $\mathcal{K}$ is $\odot$-continuous for any $\odot$.*

Proof. By contradiction, let there be a Horn clause $P \Rightarrow (t \approx t')$ and $e \in [0,1)$ such that for all $d \in [0,1)$ we have $e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} > \|d \odot P \Rightarrow (t \approx t')\|_{\mathcal{K}}$. Therefore, for each $d \in [0,1)$ there is $\boldsymbol{M}_d \in \mathcal{K}$ and a valuation $v_d : X \longrightarrow M_d$ with $e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} > \|d \odot P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_d,v_d}$. Since

$$\|d \odot P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_d,v_d} \neq 1,$$

we have $(d \odot P)(s,s') \leq \|s \approx s'\|_{\boldsymbol{M}_d,v_d}$ for each $s, s' \in T(X)$, and

$$\|d \odot P \Rightarrow (t \approx t')\|_{\boldsymbol{M}_d,v_d} = \|t \approx t'\|_{\boldsymbol{M}_d,v_d}.$$

As a consequence,

$$e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} > \|t \approx t'\|_{\boldsymbol{M}_d,v_d}. \tag{17}$$

Put

$$I = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$$

and let F be the Fréchet filter over I. For the reduced product $\prod_F \boldsymbol{M}_d$ of $\{\boldsymbol{M}_d \mid d \in I\}$ modulo F one can consider a valuation $w : X \longrightarrow (\prod_{d\in I} M_d)/\theta_F$ such that $w(x) = [v(x)]_{\theta_F}$, where $v : X \longrightarrow \prod_{d\in I} M_d$ is the induced mapping satisfying $(v(x))(d) = v_d(x)$ for all $x \in X$, and $d \in I$. Since $\odot$ is a continuous t-norm, we get

$$\begin{aligned} \|s \approx s'\|_{\prod_F \boldsymbol{M}_d,w} &= \bigvee_{Z\in F} \bigwedge_{d\in Z}\|s \approx s'\|_{\boldsymbol{M}_d,v_d} \\ &\geq \bigvee_{Z\in F} \bigwedge_{d\in Z}(d \odot P)(s,s') \\ &= \bigvee_{Z\in F} \bigwedge_{d\in Z}(d \odot P(s,s')) \\ &= \bigvee_{Z\in F}(P(s,s') \odot \textstyle\bigwedge_{d\in Z} d) \\ &= P(s,s') \odot \textstyle\bigvee_{Z\in F} \bigwedge_{d\in Z} d \\ &= P(s,s') \odot \textstyle\bigvee\{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \\ &= P(s,s') \odot 1 = P(s,s') \end{aligned}$$

for all $s, s' \in T(X)$. Hence, we have just shown $\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_d,w} = \|t \approx t'\|_{\prod_F \boldsymbol{M}_d,w}$. If we use (17) together with the facts that $e \neq 1$ and $e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} \neq 0$, we get

$$\|t \approx t'\|_{\prod_F \boldsymbol{M}_d,w} = \bigvee_{Z\in F} \bigwedge_{d\in Z}\|t \approx t'\|_{\boldsymbol{M}_d,v_d} \leq e \odot \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} < \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}.$$

By the latter inequality, we can conclude

$$\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_d} \leq \|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_d,w} = \|t \approx t'\|_{\prod_F \boldsymbol{M}_d,w} < \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}.$$

That is, we have $\|P \Rightarrow (t \approx t')\|_{\prod_F \boldsymbol{M}_d} < \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$ which contradicts the fact that $\prod_F \boldsymbol{M}_d \in \mathcal{K}$. □

Putting Theorem 6.4, Theorem 6.6, and Theorem 6.7 together, we obtain the following:

Corollary 6.8 *$\mathcal{K}$ is a $\odot$-continuous Horn class iff $\mathcal{K}$ is closed under isomorphic images, subalgebras, and reduced products.*

Remark 6.9

(i) As a consequence, a Horn class is $\odot_1$-continuous iff it is $\odot_2$-continuous. In other words, the notion of $\odot$-continuity of Horn classes does depend on the chosen $\odot$.

(ii) This is a note for readers familiar with [5, 6]. The characterization of quasivarieties presented in [6] is restricted to residuated lattices whose lattice part is a so-called complete Noetherian lattice [7]. Such a subclass of structures of truth degrees includes e. g. all finite residuated lattices but does not include residuated lattices on $[0, 1]$ given by left-continuous t-norms. In [6], the Horn classes were characterized as the classes of $\boldsymbol{L}$-algebras closed under isomorphic images, subalgebras, and safe reduced products [19] (i. e. only particular reduced products given by so-called safe filters). The present approach uses unrestricted reduced products, i. e. we take into consideration also reduced products given by filters which are not safe. Also note that the characterization of quasivarieties as continuous Horn classes can be proven alternatively using direct limits of weak direct families [19] instead of reduced products. The proof is left to the reader.

7 Continuous Horn theories

In Section 6 we investigated continuity of model classes of $\boldsymbol{L}$-algebras and showed that it can be used to prove a type of quasivariety theorem. We are now going to show that the notion of continuity can also be used to establish completeness of FHL which uses residuated lattices on the unit interval as the structures of truth degrees. Note that the result presented in this section does not have an analogy in [20]. Since we consider a fixed structure of truth degrees and define degrees of semantic consequence, we wish to prove a completeness theorem in Pavelka-style. In more detail, we wish to have a suitably defined degree of provability $|\cdot|_\Sigma$ which is equal to the degree of semantic consequence $\|\cdot\|_\Sigma$. In the sequel, we first recall all necessary syntactic notions of FHL, then we formulate the continuity on the syntactic level and use it to prove the completeness. The proofs in this section require some familiarity with [5].

For each $P \Rightarrow (t \approx t')$ let $\mathrm{Supp}(P)$ denote the support of P, i. e. $\mathrm{Supp}(P) = \{\langle s, s'\rangle \mid P(s, s') \neq 0\}$. Inference rules of FHL are defined on truth-weighted Horn clauses. Among all the rules introduced in [5], a crucial role is played by that of *monotony*:

(Mon) From $\langle P \Rightarrow (t \approx t'), b\rangle$ and $\langle Q \Rightarrow (s_1 \approx s_1'), a_1\rangle$ and … and $\langle Q \Rightarrow (s_n \approx s_n'), a_n\rangle$ infer $\langle Q \Rightarrow (t \approx t'), b\rangle$ if $\mathrm{Supp}(P) \subseteq \{\langle s_1, s_1'\rangle, \ldots, \langle s_n, s_n'\rangle\}$ and $P(s_1, s_1') \leq a_1$ and … and $P(s_n, s_n') \leq a_n$.

Given an $\boldsymbol{L}$-set Σ of Horn clauses, a *proof of* $\langle P \Rightarrow (t \approx t'), a\rangle$ *from* Σ is any sequence $\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_l, a_l\rangle$, where φ_l is $P \Rightarrow (t \approx t')$, $a_l = a$, and for each $i = 1, \ldots, l$ we have either $a_i = \Sigma(\varphi_i)$ or there is an n-ary deduction rule R of FHL [5] such that $\langle \varphi_i, a_i\rangle$ is inferred by using R applied on some $\langle \varphi_{i_1}, a_{i_1}\rangle, \ldots, \langle \varphi_{i_n}, a_{i_n}\rangle$, where $i_1, \ldots, i_n < i$. $\langle P \Rightarrow (t \approx t'), b\rangle$ is *provable from* Σ, denoted by $\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle$, if there is a proof of $\langle P \Rightarrow (t \approx t'), b\rangle$ from Σ. For every Horn clause $P \Rightarrow (t \approx t')$ we define a *degree* $|P \Rightarrow (t \approx t')|_\Sigma$ *of provability of* $P \Rightarrow (t \approx t')$ *from* Σ by $|P \Rightarrow (t \approx t')|_\Sigma = \bigvee\{a \in L \mid \Sigma \vdash \langle P \Rightarrow (t \approx t'), a\rangle\}$.

Definition 7.1 $\boldsymbol{L}$-set Σ of Horn clauses is called $\odot$*-continuous* if for each Horn clause $P \Rightarrow (t \approx t')$ and each $e \in [0, 1)$ there is $d \in [0, 1)$ such that

if $\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle$, then there is $b_e \in [e \odot b, 1]$ such that $\Sigma \vdash \langle d \odot P \Rightarrow (t \approx t'), b_e\rangle$.

Theorem 7.2 (Completeness) *Let* Σ *be a* $\odot$*-continuous* $\boldsymbol{L}$*-set of Horn clauses. Then*

$$|P \Rightarrow (t \approx t')|_\Sigma = \|P \Rightarrow (t \approx t')\|_\Sigma \tag{18}$$

for each Horn clause $P \Rightarrow (t \approx t')$.

P r o o f. According to [5, Theorem 12], it suffices to show that if

$$P(s, s') \leq |Q \Rightarrow (s \approx s')|_\Sigma$$

for all $s, s' \in T(X)$, then $|P \Rightarrow (t \approx t')|_\Sigma \leq |Q \Rightarrow (t \approx t')|_\Sigma$. The equality (18) will then be a consequence of [5, Theorem 8, Theorem 11, and Theorem 12].

Let $P(s, s') \leq |Q \Rightarrow (s \approx s')|_\Sigma$ for all $s, s' \in T(X)$. It suffices to check that if

$$\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle,$$

then $b \leq |Q \Rightarrow (t \approx t')|_\Sigma$. Thus, let $\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle$. Since Σ is $\odot$-continuous for each $e \in [0, 1)$ there is $d \in [0, 1)$ such that $\Sigma \vdash \langle d \odot P \Rightarrow (t \approx t'), b_e\rangle$, where $b_e \geq e \odot b$. If $P(s, s') > 0$, then we have

$$(d \odot P)(s, s') < P(s, s') \leq |Q \Rightarrow (s \approx s')|_\Sigma.$$

Therefore, for each $s, s' \in T(X)$ there exists a truth degree $a_{s,s'} \in L$ such that $\Sigma \vdash \langle Q \Rightarrow (s \approx s'), a_{s,s'}\rangle$ and $a_{s,s'} \geq (d \odot P)(s, s')$. Hence, by (Mon), one has

$$\Sigma \vdash \langle Q \Rightarrow (t \approx t'), b_e\rangle.$$

That is, for each $e \in [0, 1)$, $\Sigma \vdash \langle Q \Rightarrow (t \approx t'), b_e\rangle$ with $b \odot e \leq b_e$. So, for each $e \in [0, 1)$,

$$b \odot e \leq |Q \Rightarrow (t \approx t')|_\Sigma,$$

i. e. $b \leq |Q \Rightarrow (t \approx t')|_\Sigma$ by Lemma 6.3(i). This gives $|P \Rightarrow (t \approx t')|_\Sigma \leq |Q \Rightarrow (t \approx t')|_\Sigma$. □

Lemma 7.3 *Let Σ be an $\boldsymbol{L}$-set of Horn clauses.*

(i) *If Σ is $\odot$-continuous, then* $\mathrm{Mod}(\Sigma)$ *is $\odot$-continuous.*

(ii) *If* $\mathrm{Mod}(\Sigma)$ *is $\odot$-continuous, then* $\mathrm{Horn}(\mathrm{Mod}(\Sigma))$ *is $\odot$-continuous.*

Proof.

(i) Let Σ be $\odot$-continuous. Take a Horn clause $P \Rightarrow (t \approx t')$. For each $e \in [0, 1)$ there is $d \in [0, 1)$ such that if $\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle$ for $b \in [0, 1]$, then $\Sigma \vdash \langle d \odot P \Rightarrow (t \approx t'), b_e\rangle$ for $b_e \geq b \odot e$. Thus, for each $b \in [0, 1]$, if $\Sigma \vdash \langle P \Rightarrow (t \approx t'), b\rangle$, then $b \odot e \leq |d \odot P \Rightarrow (t \approx t')|_\Sigma$. This immediately gives

$$e \odot |P \Rightarrow (t \approx t')|_\Sigma \leq |d \odot P \Rightarrow (t \approx t')|_\Sigma$$

which further yields $e \odot \|P \Rightarrow (t \approx t')\|_\Sigma \leq \|d \odot P \Rightarrow (t \approx t')\|_\Sigma$ by Theorem 7.2, i. e.

$$e \odot \|P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)} \leq \|d \odot P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)}.$$

That is, $\mathrm{Mod}(\Sigma)$ is $\odot$-continuous.

(ii) Let $\mathrm{Mod}(\Sigma)$ be $\odot$-continuous. Take $P \Rightarrow (t \approx t')$ and $e \in [0, 1)$. Using the $\odot$-continuity of $\mathrm{Mod}(\Sigma)$, we can take $d \in [0, 1)$ such that

$$e \odot \|P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)} \leq \|d \odot P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)}.$$

Let $\mathrm{Horn}(\mathrm{Mod}(\Sigma)) \vdash \langle P \Rightarrow (t \approx t'), b\rangle$. We have

$$b \leq |P \Rightarrow (t \approx t')|_{\mathrm{Horn}(\mathrm{Mod}(\Sigma))} \leq \|P \Rightarrow (t \approx t')\|_{\mathrm{Horn}(\mathrm{Mod}(\Sigma))} = \|P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)}$$

by soundness of FHL and $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(\mathrm{Horn}(\mathrm{Mod}(\Sigma)))$. Moreover, the $\odot$-continuity of $\mathrm{Mod}(\Sigma)$ yields

$$\begin{aligned} e \odot b &\leq e \odot \|P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)} \\ &\leq \|d \odot P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)} \\ &= \mathrm{Horn}(\mathrm{Mod}(\Sigma))(d \odot P \Rightarrow (t \approx t')). \end{aligned}$$

Put $b_e = \mathrm{Horn}(\mathrm{Mod}(\Sigma))(d \odot P \Rightarrow (t \approx t'))$. We now have that

$$\mathrm{Horn}(\mathrm{Mod}(\Sigma)) \vdash \langle d \odot P \Rightarrow (t \approx t'), b_e\rangle$$

and $b_e \in [e \odot b, 1]$, i. e. $\mathrm{Horn}(\mathrm{Mod}(\Sigma))$ is $\odot$-continuous. □

The next corollary shows the relationship between $\odot$-continuous Horn classes and $\odot$-continuous Horn theories.

Corollary 7.4 *Let Σ be an $\boldsymbol{L}$-set of Horn clauses. The following assertions are equivalent.*

(i) *Horn theory of* $\mathrm{Mod}(\Sigma)$ (*i. e. the semantic closure of* Σ) *is $\odot$-continuous.*

(ii) *Horn class of Σ is $\odot$-continuous.*

(iii) *Horn class of Σ is closed under isomorphic images, subalgebras, and reduced products.*

Remark 7.5 As an immediate consequence of Corollary 7.4 we get that the $\odot$-continuity of the semantic closure of Σ does not depend on the chosen $\odot$.

Example 7.6

1. Let $\boldsymbol{L}$ be a residuated lattice on $[0,1]$ with $\otimes = \wedge$. One can prove by induction on the length of a proof that if for each $d \in [0,1)$ and $P \Rightarrow (t \approx t')$ we have $d \odot \Sigma(P \Rightarrow (t \approx t')) \leq \Sigma(d \odot P \Rightarrow (t \approx t'))$, then Σ is $\odot$-continuous.

2. Using 1., each $\boldsymbol{L}$-set ($\otimes = \wedge$) of identities (Horn clauses of the form $\emptyset \Rightarrow (t \approx t')$) is $\odot$-continuous.

3. Let F be a type which consists of a single binary function symbol f, Σ be an $\boldsymbol{L}$-set of Horn clauses, where $\Sigma(\langle x \circ z \approx y \circ z, a\rangle \Rightarrow x \approx y) = a$ for each $a \in [0,1]$, and $\Sigma(P \Rightarrow (t \approx t')) = 0$ otherwise. If $\otimes = \wedge$, 1. gives that Σ is $\odot$-continuous. Σ can be seen as a theory of grupoids (understood as $\boldsymbol{L}$-algebras) satisfying a particular type of continuous cancellation.

4. There are, of course, $\boldsymbol{L}$-sets of Horn clauses which are not $\odot$-continuous. For instance, consider Σ such that $\Sigma(\langle x \approx y, 0.5\rangle \Rightarrow x \approx y) = 1$, and $\Sigma(P \Rightarrow (t \approx t')) = 0$ otherwise. For $e = 0.9$ there is not any $d \in [0,1)$ such that $\Sigma \vdash \langle\langle x \approx y, d \odot 0.5\rangle \Rightarrow x \approx y, b_e\rangle$ for $b_e \geq e \odot 1 = 0.9$, because for each $d \in [0,1)$ we can consider an $\boldsymbol{L}$-algebra $\boldsymbol{M} = \langle M, \approx^{\boldsymbol{M}}, \emptyset\rangle$ (of the empty type) with $M = \{\mathfrak{a}, \mathfrak{b}\}$ and $\mathfrak{a} \approx^{\boldsymbol{M}} \mathfrak{b} = d \odot 0.5$ which is a model of such Σ but for a valuation $v : X \longrightarrow M$ with $v(x) = \mathfrak{a}$ and $v(y) = \mathfrak{b}$ we have

$$\|\langle x \approx y, d \odot 0.5\rangle \Rightarrow x \approx y\|_{\boldsymbol{M},v} = d \odot 0.5 \ngeq 0.9 = b_e.$$

Therefore, soundness of FHL gives that Σ is not $\odot$-continuous.

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