# Fuzzy attribute logic over complete residuated lattices

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We present a logic, called fuzzy attribute logic, for reasoning about formulas describing particular attribute dependencies. The formulas are of a form $A \Rightarrow B$ where $A$ and $B$ are collections of attributes. Our formulas can be interpreted in two ways. First, in data tables with entries containing degrees to which objects (table rows) have attributes (table columns). Second, in database tables where each domain is equipped with a similarity relation assigning a degree of similarity to any pair of domain elements. We assume that the scale of degrees is equipped with fuzzy logical connectives and forms an arbitrary complete residuated lattice. This covers many structures used in fuzzy logic applications as well as structures used in formal systems of fuzzy logic. If the scale contains only two degrees, 0 (falsity) and 1 (truth), two well-known calculi become particular cases of our logic. Namely, with the first interpretation, our logic coincides with attribute logic used in formal concept analysis; with the second interpretation, our logic coincides with Armstrong system for reasoning about functional dependencies.

We prove completeness of fuzzy attribute logic over arbitrary complete residuated lattices in two versions. First, in the ordinary style, completeness asserts that a formula $A \Rightarrow B$ is entailed by a collection $T$ of formulas iff $A \Rightarrow B$ is provable from $T$. Second, in the graded style, completeness asserts that a degree to which $A \Rightarrow B$ is entailed by a collection $T$ of formulas equals a degree to which $A \Rightarrow B$ is provable from $T$.

## 1 Introduction

Rules $A \Rightarrow B$, e.g. $\{\text{prime}, {>}2\} \Rightarrow \{\text{odd}\}$, $\{\text{flight No.}\} \Rightarrow \{\text{departure time}, \text{arrival time}\}$, are perhaps the most important means for description of attribute dependencies in symbolic data. These rules are used in various areas. In formal concept analysis (Ganter and Wille, 1999), the rules are called attribute implications and are interpreted in tables describing objects and attributes. Entries in a table say whether or not a particular object has a particular attribute. Then, $A \Rightarrow B$ is true in a table if each object having all attributes from $A$ has also all attributes from $B$, cf. $\{\text{prime}, {>}2\} \Rightarrow \{\text{odd}\}$. Another area where these rules are used is data mining. The rules are called association rules and their interpretation is essentially the same as in formal concept analysis except that two additional parameters, called support and confidence, are used to tell which rules are interesting, see (Zhang and Zhang, 2002) for details. A different interpretation of rules $A \Rightarrow B$ is used in relational databases. The rules are called functional dependencies and they are used for management of data redundancy and for database design, see (Maier, 1983). Functional dependencies are interpreted in database tables and $A \Rightarrow B$ being true in a table means that any two tuples (rows) of the table which have the same values in all attributes from $A$ have also the same values in all attributes from $B$, cf. $\{\text{flight No.}\} \Rightarrow \{\text{departure time}, \text{arrival time}\}$.

It has been repeatedly stressed that current methods of data management have only very limited capability of dealing with uncertainty in data, see e.g (Abiteboul *et al.*, 2005). Several calculi have been developed for treatment of various forms of uncertainty in recent years. Among them, fuzzy logic plays an important role. In our previous papers, we studied several aspects of rules $A \Rightarrow B$ from the point of view of fuzzy logic, see (Bělohlávek *et al.*, 2004; Bělohlávek and Vychodil, 2005a,b,c, 2006a,b). We introduced two interpretations of formulas $A \Rightarrow B$. First, in data tables with entries containing degrees to which objects (table rows) have attributes (table columns). This interpretation generalizes the one used in formal concept analysis which results as a special case when our scale of truth degrees contains just two degrees 0 (falsity) and 1 (truth). Second, in database tables where each domain is equipped with a similarity relation assigning a degree of similarity to any pair of domain elements. Again, the ordinary database interpretation results as a special case when our scale contains just 0 and 1 and when the domain

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similarities are identity relations. We studied problems motivated in data analysis (e.g., a feasible description and computation of non-redundant bases from data tables), relationship to the ordinary setting, and database issues. In addition to that, we introduced fuzzy attribute logic, see (Bělohlávek and Vychodil, 2005a,c), i.e. a logical calculus for reasoning with formulas $A \Rightarrow B$. However, up to now, completeness of fuzzy attribute logic is restricted to finite structures of truth degrees. As a result, important structures like the unit interval $[0,1]$ elude completeness.

The current paper presents completeness of fuzzy attribute logic over arbitrary complete residuated lattices, including thus $[0,1]$ equipped with an arbitrary left-continuous t-norm. We prove two types of completeness theorems: ordinary-style completeness (provability is a bivalent notion) and graded-style completeness (provability is a matter of degree). Ordinary-style completeness asserts that a formula $A \Rightarrow B$ is entailed by a collection $T$ of formulas iff $A \Rightarrow B$ is provable from $T$. Graded-style completeness asserts that a degree to which $A \Rightarrow B$ is entailed by $T$ equals a degree to which $A \Rightarrow B$ is provable from $T$.

## 2 Preliminaries

Fuzzy attribute logic is developed over complete residuated lattices with truth-stressing hedges (shortly, hedges). A complete residuated lattice with a hedge, which is our basic structure of truth degrees, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of $L$, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge $^*$ satisfies

$$1^* = 1, \tag{2}$$

$$a^* \leq a, \tag{3}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{4}$$

$$a^{**} = a^*, \tag{5}$$

for all $a, b \in L$. Elements $a$ of $L$ are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge $^*$ is a (truth function of) logical connective "very true", see (Hájek, 1998, 2001). Properties (2)–(5) have natural interpretations, e.g. (4) can be read: "if $a \rightarrow b$ is very true and if $a$ is very true, then $b$ is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations, i.e. $\otimes$ and $\rightarrow$ satisfying (1), on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a+b-1, 0), \\ a \rightarrow b &= \min(1-a+b, 1), \end{aligned} \tag{6}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a,b), \\ a \rightarrow b &= \begin{cases} 1 \text{ if } a \leq b, \\ b \text{ otherwise}, \end{cases} \end{aligned} \tag{7}$$

$$\text{Goguen (product):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 \text{ if } a \leq b, \\ \frac{b}{a} \text{ otherwise}. \end{cases} \end{aligned} \tag{8}$$

Complete residuated lattices include also finite structures of truth degrees. For instance, we can take a finite subset $L \subseteq [0,1]$ that is closed under Łukasiewicz or Gödel operations. If we take $L = \{0,1\}$, we obtain this way the two-element Boolean algebra (structure of truth degrees of classical logic). Two boundary cases of hedges are (i) identity, i.e. $a^* = a$ $(a \in L)$; (ii) globalization (Takeuti and Titani, 1987):

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Given $\mathbf{L}$ which serves as a structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) $A$ in universe $U$ is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which $u$ belongs to $A$". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in $U$. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in $U$ such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. For $a \in L$ and $A \in \mathbf{L}^U$, we define $\mathbf{L}$-sets $a \otimes A$ ($a$-multiple of $A$) and $a \to A$ ($a$-shift of $A$) by $(a \otimes A)(u) = a \otimes A(u)$, $(a \to A)(u) = a \to A(u)$ $(u \in U)$. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A,B) = \textstyle\bigwedge_{u \in U}(A(u) \to B(u)),$$

which generalizes the classical subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents the degree to which $A$ is a subset of $B$. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$, i.e. if $A$ is fully included in $B$. We have $A \subseteq B$ iff, for each $u \in U$, $A(u) \leq B(u)$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found, e.g., in monographs (Bělohlávek, 2002; Hájek, 1998).

## 3 Fuzzy attribute logic: soundness and completeness

### 3.1 *Validity and semantic entailment*

In this section we recall basic notions of fuzzy attribute logic (FAL). More details can be found in (Bělohlávek *et al.*, 2004; Bělohlávek and Vychodil, 2005b, 2006a).

In what follows, we let $Y$ be a *finite set of attributes,* each $y \in Y$ will be called an *attribute. Fuzzy attribute implication* (*over attributes* $Y$) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ ($A$ and $B$ are fuzzy sets of attributes). Fuzzy attribute implications (FAIs) are the formulas of fuzzy attribute logic. In order to consider validity (truth) of FAIs, we introduce a semantic component in which we evaluate FAIs and their formal interpretation. The intuitive meaning we wish to give to $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from $A$, then it has also all attributes from $B$". Formally, for an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is true in* $M$ by

$$||A \Rightarrow B||_M = S(A,M)^* \to S(B,M), \tag{10}$$

where $S(\cdots)$ denote subsethood degrees, and $\to$ and $^*$ are operations of a "fuzzy conjunction" and a hedge of a complete residuated lattice $\mathbf{L}$, see Section 2. The degree $||A \Rightarrow B||_M$ can be understood as follows: if $M$ (semantic component) represents presence of attributes of some object, i.e. $M(y)$ is truth degree to which "the object has the attribute $y \in Y$", then $||A \Rightarrow B||_M$ is the truth degree of "if the object has all attributes from $A$, then it has all attributes from $B$", which corresponds to the desired interpretation of $A \Rightarrow B$. Note also that the hedge $^*$ serves as a modifier of interpretation of $A \Rightarrow B$, see (Bělohlávek *et al.*, 2004; Bělohlávek and Vychodil, 2005b, 2006a) for details. For instance, if $^*$ is globalization then $||A \Rightarrow B||_M = 1$ (i.e. $A \Rightarrow B$ is true in $M$) means that if $A \subseteq M$ then $B \subseteq M$, i.e. only full subsethood is taken into account. If $^*$ is identity then $||A \Rightarrow B||_M = 1$ means $S(A,M) \leq S(B,M)$, i.e. any subsethood degree is taken into account.

Let now $\langle X, Y, I \rangle$ be a data table with fuzzy attributes, i.e. $X$ be a finite set of objects, $Y$ be a finite set of attributes, and $I$ be an $\mathbf{L}$-relation between $X$ and $Y$ assigning to any object $x \in X$ and any attribute $y \in Y$ a degree $I(x,y) \in L$ to which $x$ has $y$. For any $x \in X$, denote by $M_x$ a fuzzy set of attributes of $x$, i.e. $M_x(y) = I(x,y)$ for each attribute $y \in Y$. Then, one can define a degree $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ to which

$A \Rightarrow B$ is true in the data table $\langle X, Y, I \rangle$ by

$$||A \Rightarrow B||_{\langle X,Y,I \rangle} = \bigwedge_{x \in X} ||A \Rightarrow B||_{M_x}.$$

Note that $M_x$ can be seen as a table row corresponding to $x$ and thus $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ is a degree to which "$A \Rightarrow B$ is true in each row of the table $\langle X, Y, I \rangle$".

Let $T$ be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model of* $T$ if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of $T$ is denoted by $\mathrm{Mod}(T)$. A *degree* $||A \Rightarrow B||_T \in L$ *to which* $A \Rightarrow B$ *semantically follows from* $T$ is defined by

$$||A \Rightarrow B||_T = \bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M. \tag{11}$$

Described verbally, $||A \Rightarrow B||_T$ is defined as a degree to which "$A \Rightarrow B$ is true in each model of $T$". Hence, $||\cdots||_T$ defined by (11) represents a degree of semantic entailment from $T$. In further sections we will be interested in syntactic characterizations of $||\cdots||_T$.

*Remark 1* The database interpretation of fuzzy attribute implication mentioned in Section 1 will not be described in this paper. Namely, both of the interpretations have the same concept of semantic entailment. That is a degree $||A \Rightarrow B||_T$ defined by (11) equals a degree to which $A \Rightarrow B$ semantically follows from $T$ when using the database interpretation. Therefore, a logical calculus which is complete (either in ordinary or in graded style) with respect to one of these interpretations is also complete with respect to the other interpretation. For details we refer to (Bělohlávek and Vychodil, 2006a,b).

### 3.2 *Provability and syntactic entailment*

This section introduces a deductive system for fuzzy attribute logic and a particular notion of provability which generalize the ones presented in (Bělohlávek and Vychodil, 2005c). The generalized notions will allow us to prove various completeness theorems (in Section 3.3) over each complete residuated lattice with hedge taken as the structure of truth degrees.

Deductive systems for FAL presented in (Bělohlávek and Vychodil, 2005a,c) are based on *deduction rules* of the form "from $\varphi_1, \ldots, \varphi_n$ infer $\varphi$", where $n$ is a nonnegative integer and $\varphi, \varphi_i$ $(i \in I)$ are (schemas for) FAIs. Such deduction rules are to be understood as usual: having rule "from $\varphi_1, \ldots, \varphi_n$ infer $\varphi$" and FAIs which are of the form of FAIs in the input part (the part preceding "infer") of the rule, the rule allows us to infer (in one step) the corresponding FAI in the output part (the part following "infer") of the rule. Each nullary rule, i.e. rule where $n = 0$, is considered as an axiom the output part of which can be inferred in one step.

Deductive system of FAL which has been introduced in (Bělohlávek and Vychodil, 2005c) uses the following rules.

(Ax) infer $A \cup B \Rightarrow A$,
(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,
(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Notice that (Ax) is a nullary rule (axiom) which says that each $A \cup B \Rightarrow A$ $(A, B \in \mathbf{L}^Y)$ is inferred in one step. The rules are inspired by Armstrong-like axioms, see (Armstrong, 1974) and also (Maier, 1983) for a good overview.

A fuzzy attribute implication $A \Rightarrow B$ is called *provable from a set* $T$ *of FAIs using a set* $\mathcal{R}$ *of deduction rules* if there is a sequence $\varphi_1, \ldots, \varphi_n$ of fuzzy attribute implications such that $\varphi_n$ is $A \Rightarrow B$ and for each $\varphi_i$ we either have $\varphi_i \in T$ or $\varphi_i$ is inferred (in one step) from some of the preceding formulas using some deduction rule from $\mathcal{R}$ (i.e., $\mathcal{R}$ contains a rule "from $\psi_1, \ldots, \psi_k$ infer $\varphi_i$" where each of $\psi_1, \ldots, \psi_k$ is among $\varphi_1, \ldots, \varphi_{i-1}$). If $\mathcal{R}$ consists of (Ax)–(Mul), we say just "$A \Rightarrow B$ is provable from $T$" instead of "$A \Rightarrow B$ is provable from $T$ using ..." and denote this fact by $T \vdash A \Rightarrow B$.

A deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer $\varphi$" is said to be *sound* if $\mathrm{Mod}(\{\varphi_1, \ldots, \varphi_n\}) \subseteq \mathrm{Mod}(\{\varphi\})$. A deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer $\varphi$" ($\varphi_i, \varphi$ are FAIs) is said to be *derivable* (*from a set* $\mathcal{R}$ *of deduction* *rules*) if $\varphi$ is provable from $\{\varphi_1, \dots, \varphi_n\}$ (using $\mathcal{R}$). A proof of the following observation follows from the the ordinary case, see (Bělohlávek and Vychodil, 2005c).

LEMMA 3.1 *The following deduction rules are derivable from* (Ax) *and* (Cut):

(Add) *from* $A \Rightarrow B$ *and* $A \Rightarrow C$ *infer* $A \Rightarrow B \cup C$,

(Pro) *from* $A \Rightarrow B \cup C$ *infer* $A \Rightarrow B$,

(Tra) *from* $A \Rightarrow B$ *and* $B \Rightarrow C$ *infer* $A \Rightarrow C$,

*for each* $A, B, C, D \in \mathbf{L}^Y$. □

In (Bělohlávek and Vychodil, 2005a,c), we showed that given a finite residuated lattice with a hedge, we have that $T \vdash A \Rightarrow B$ iff $||A \Rightarrow B||_T = 1$. Described verbally, FAIs which are semantically entailed from $T$ to degree 1 (fully entailed) are exactly the FAIs which are provable from $T$ using (Ax), (Cut), and (Mul). We now extend the deductive system of FAL so that we will be able to prove this claim (and much more) for *any* complete residuated lattice with a hedge.

The present deductive system will be extended by the following infinitary deduction rule which can have infinitely many FAIs in the input part (part preceding "infer"):

(Add$_\omega$) from $A \Rightarrow B_i$ $(i \in I)$ infer $A \Rightarrow \bigcup_{i \in I} B_i$,

where $\{A \Rightarrow B_i \,|\, i \in I\}$ is an $I$-indexed set of FAIs. Thus, (Add$_\omega$) is of the form "from $\varphi_i$ $(i \in I)$ infer $\varphi$". Rules of this form will be called *$\omega$-deduction rules* (shortly, *$\omega$-rules*). Clearly, (Ax), (Cut), and (Mul) can also be seen as $\omega$-rules, where $I$ is finite. Note that infinitary rules are widely used in logic and computer science. For instance, they are used in universal algebra (Wechler, 1992) as well as fuzzy logic in narrow sense (e.g., in TT∀, see (Hájek, 1998)). From now on, assume that we use a deductive system which consists of (Ax), (Cut), (Mul), and (Add$_\omega$).

In order to use $\omega$-rules we switch from proofs considered as finite sequences of FAIs to $\omega$-proofs which will be defined as certain labeled infinitely branching rooted (directed) trees with finite depth (Grimaldi, 2004; Wechler, 1992). Each tree will be denoted by $\mathcal{T} = \langle l, St \rangle$, where $l$ is a label in the root of $\mathcal{T}$, and $St$ is a set of subtrees of $\mathcal{T}$: $St = \{\mathcal{T}_v \,|\, \mathcal{T}_v$ is a subtree at $v$, and $v$ is a direct descendant of the root of $\mathcal{T}\}$, see (Grimaldi, 2004). Leaf nodes are thus denoted $\mathcal{T} = \langle l, \emptyset \rangle$.

Given a set $T$ of fuzzy attribute implications, we define an *$\omega$-proof from $T$* (*using deductive system* $\mathcal{R}$) as follows: (i) for each $A \Rightarrow B \in T$, tuple $\mathcal{T} = \langle A \Rightarrow B, \emptyset \rangle$ is an $\omega$-proof from $T$ (using $\mathcal{R}$); (ii) if $\mathcal{T}_i = \langle \varphi_i, \dots \rangle$ $(i \in I)$ are $\omega$-proofs from $T$ (using $\mathcal{R}$) and "from $\varphi_i$ $(i \in I)$ infer $\varphi$" is a rule in $\mathcal{R}$ then $\mathcal{T} = \langle \varphi, \{\mathcal{T}_i \,|\, i \in I\} \rangle$ is an $\omega$-proof from $T$ (using $\mathcal{R}$). $A \Rightarrow B$ is called $\omega$-provable from $T$ (using $\mathcal{R}$), written $T \vdash_\omega A \Rightarrow B$ $(T \vdash_{\omega,\mathcal{R}} A \Rightarrow B)$, if there is an $\omega$-proof $\mathcal{T}$ from $T$ (using $\mathcal{R}$) such that $\mathcal{T} = \langle A \Rightarrow B, \dots \rangle$ (i.e., if $A \Rightarrow B$ is the label in the root node of $\mathcal{T}$). One can introduce notions of soundness and $\omega$-derivability of $\omega$-rules as in the previous setting.

The following assertion shows that, considering only (Ax), (Cut), and (Mul), provability coincides with $\omega$-provability.

THEOREM 3.2 $T \vdash A \Rightarrow B$ *iff* $A \Rightarrow B$ *is $\omega$-provable from* $T$ *using* (Ax), (Cut), *and* (Mul).

*Proof* "⇒": Let $T \vdash A \Rightarrow B$. Thus, there is a sequence of FAIs $\varphi_1, \dots, \varphi_n$ which is a proof of $A \Rightarrow B$ from $T$. By induction, we can check that for each $\varphi_i$ $(i = 1, \dots, n)$ there is an $\omega$-proof of $\varphi_j$.

"⇐": Let $A \Rightarrow B$ be $\omega$-provable from $T$ using (Ax), (Cut), and (Mul). Thus, there is an $\omega$-proof $\mathcal{T}$ using (Ax), (Cut), and (Mul) which is finite. Now, one can construct a sequence of FAIs by depth-first traversing $\mathcal{T}$ and listing all labels in post-order. This sequence is a proof of $A \Rightarrow B$ from $T$ (easy to check). □

The following assertion shows that if we use finite structures of truth degrees, $\omega$-provability yields exactly the same as (the usual) provability. In this sense, our extension of the deductive system of FAL is conservative with respect to the previous results.

THEOREM 3.3 *Let* $\mathbf{L}$ *be a finite residuated lattice with hedge. Then, for each set* $T$ *of FAIs and for each* $A \Rightarrow B$,

$$T \vdash_\omega A \Rightarrow B \quad \textit{iff} \quad T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1.$$

*Proof* It suffices to show $T \vdash_\omega A \Rightarrow B$ iff $T \vdash A \Rightarrow B$. The rest is a consequence of the completeness of FAL for finite structures of truth degrees, see (Bělohlávek and Vychodil, 2005a).

"$\Rightarrow$": The crucial observation here is that since both $L$ and $Y$ are finite, we can equivalently replace (Add$_\omega$) by a collection of rules

(Add$_n$) from $A \Rightarrow B_1, \ldots, A \Rightarrow B_n$ infer $A \Rightarrow B_1 \cup \cdots \cup B_n$,

for each $n$ being a nonnegative integer. Namely, due to finiteness of $L$ and $Y$, there is only a finite number of fuzzy sets in $Y$ (i.e. $\mathbf{L}^Y$ is finite) and, therefore, only a finite number of FAIs. It is clear from Lemma 3.1 that each (Add$_n$) is derivable from (Ax), (Cut), and (Mul). Hence, if $T \vdash_\omega A \Rightarrow B$, then there is a finite $\omega$-proof $\mathcal{T}$ of $A \Rightarrow B$ from $T$ which uses only (Ax), (Cut), and (Mul). Now apply Theorem 3.2.

"$\Leftarrow$": Follows from Theorem 3.2. □

*Remark 2* For general $\mathbf{L}$, (Add$_\omega$) is not $\omega$-derivable from (Ax)–(Mul). Indeed, let $L = [0,1]$ and $Y = \{y\}$. It suffices to show that FAI $\{\} \Rightarrow \{y\}$ is not $\omega$-provable from a set $T = \{\{\} \Rightarrow \{{}^a/y\} \,|\, a < 1\}$ using (Ax)–(Mul). By contradiction, let $\mathcal{T}$ be an $\omega$-proof of $\{\} \Rightarrow \{y\}$ from $T$ which uses only rules (Ax), (Cut), and (Mul). By Theorem 3.2, $T \vdash \{\} \Rightarrow \{y\}$. By a standard argument, there is a finite subset $T'$ of $T$ such that $T' \vdash \{\} \Rightarrow \{y\}$. Now, for $b = \bigvee\{a \in L \,|\, \{\} \Rightarrow \{{}^a/y\} \in T'\}$ we have $b < 1$. Since $L = [0,1]$, one can take a model $M \in \mathbf{L}^Y$ of $T'$ such that $b < M(y) < 1$. For $M$, we have $||\{\} \Rightarrow \{y\}||_M = M(y)$. The latter observation yields $||\{\} \Rightarrow \{y\}||_{T'} < 1$. Since (Ax)–(Mul) are sound (Bělohlávek and Vychodil, 2005c), we obtain $T' \nvdash \{\} \Rightarrow \{y\}$, a contradiction.

### 3.3 *Soundness and completeness*

We now prove completeness of FAL in two versions. First, we show that FAIs which are $\omega$-provable from $T$ are those which semantically follow from $T$ to degree 1 (full truth). Recall that we assume that $\mathbf{L}$ is a complete residuated lattice with a hedge.

THEOREM 3.4 (Add$_\omega$) *is sound. Moreover, for each set $T$ of FAIs and each $A \Rightarrow B$ we have: if $T \vdash_\omega A \Rightarrow B$, then $||A \Rightarrow B||_T = 1$.*

*Proof* Let $T = \{A \Rightarrow B_i \,|\, i \in I\}$ be a set of FAIs, and let $M \in \mathrm{Mod}(T)$. Then, for each $i \in I$, $||A \Rightarrow B_i||_M = 1$, which is equivalent to $S(A,M)^* \leq S(B_i, M)$. Thus, $S(A,M)^* \leq \bigwedge_{i \in I} S(B_i, M)$, which further gives $S(A,M)^* \leq S(\bigcup_{i \in I} B_i, M)$, i.e. $||A \Rightarrow \bigcup_{i \in I} B_i||_M = 1$, showing that (Add$_\omega$) is sound. The rest can be proved by induction using the fact that (Ax), (Cut), and (Mul) are sound (Bělohlávek and Vychodil, 2005c). □

LEMMA 3.5 *Let $T$ be a set of FAIs. If $T \nvdash_\omega A \Rightarrow B$, then there is a model $M \in \mathrm{Mod}(T)$ such that $||A \Rightarrow B||_M \neq 1$.*

*Proof* Let $T \nvdash_\omega A \Rightarrow B$. Put $M = \bigcup\{C \in \mathbf{L}^Y \,|\, T \vdash_\omega A \Rightarrow C\}$. We have that $T \vdash_\omega A \Rightarrow M$. Indeed, for each $T \vdash_\omega A \Rightarrow C$, there is an $\omega$-proof $\mathcal{T}_C$ of $A \Rightarrow C$ from $T$. Hence, using (Add$_\omega$), $\mathcal{T} = \langle A \Rightarrow M, \{\mathcal{T}_C \,|\, T \vdash_\omega A \Rightarrow C\}\rangle$ is an $\omega$-proof of $A \Rightarrow M$ from $T$, showing $T \vdash_\omega A \Rightarrow M$.

Now, it suffices to check that (i) $M \in \mathrm{Mod}(T)$ and (ii) $||A \Rightarrow B||_M \neq 1$.

Ad (i): Let $C \Rightarrow D \in T$. We need to show $||C \Rightarrow D||_M = 1$, i.e. $S(C,M)^* \leq S(D,M)$ which is equivalent to $S(C,M)^* \otimes D \subseteq M$. Hence, it is sufficient to show that $T \vdash_\omega A \Rightarrow S(C,M)^* \otimes D$ which is indeed true. In more detail, we have that

$T \vdash_\omega A \Rightarrow M$ [see above],

$T \vdash_\omega M \Rightarrow S(C,M)^* \otimes C$ [instance of (Ax)],

$T \vdash_\omega S(C,M)^* \otimes C \Rightarrow S(C,M)^* \otimes D$ [by (Mul) on $C \Rightarrow D$],

Thus, $T \vdash_\omega A \Rightarrow S(C,M)^* \otimes D$ follows by (Tra).

Ad (ii): By contradiction, suppose $||A \Rightarrow B||_M = 1$. Since $A \subseteq M$ by definition, we then get $1 = ||A \Rightarrow B||_M = S(A,M)^* \to S(B,M) = 1 \to S(B,M) = S(B,M)$, i.e. $B \subseteq M$. Since $T \vdash_\omega A \Rightarrow M$, (Pro) would give $T \vdash_\omega A \Rightarrow B$, a contradiction. □

We now have the ordinary-style completeness for fuzzy attribute logic over arbitrary complete residuated lattices:

THEOREM 3.6 *Let $T$ be a set of FAIs. Then*

$$T \vdash_\omega A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1.$$

*Proof* The "$\Rightarrow$"-part follows from Theorem 3.4. For the "$\Leftarrow$"-part, observe that due to Lemma 3.5, if $T \not\vdash_\omega A \Rightarrow B$, then there is $M \in \mathrm{Mod}(T)$ such that $||A \Rightarrow B||_M \neq 1$, i.e. $||A \Rightarrow B||_T \neq 1$, which finishes the proof. □

Ordinary-style completeness says that a FAI is provable from $T$ iff it is entailed by $T$ in degree 1. Therefore, ordinary-style completeness can be seen as a syntactic description of semantic entailment in degree 1. However, semantic entailment is a graded notion and there is a natural question whether an arbitrary degree $||A \Rightarrow B||_T$ (possibly different from 1) can be described syntactically using a suitable notion of a proof. To this end, we adopt the concept of a degree of provability which was introduced for the first time in (Pavelka, 1979), see also (Gerla, 2001) for a thorough treatment of this concept. Suppose now that $T$ is a fuzzy set of FAIs. Note that this is natural when considering degrees of entailment and degrees of provability. Namely, for a FAI $A \Rightarrow B$, a degree $T(A \Rightarrow B)$ to which $A \Rightarrow B$ belongs to $T$ can be seen as a degree to which a validity of $A \Rightarrow B$ has been established and thus to which we can rely on $A \Rightarrow B$ when proving from $T$. For a fuzzy set $T$ of FAIs, define an ordinary set $\mathrm{c}(T)$ of FAIs by

$$\mathrm{c}(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in \mathbf{L}^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \emptyset\}.$$

Therefore, for each FAI $A \Rightarrow B$ such that $T(A \Rightarrow B) \otimes B \neq \emptyset$, $\mathrm{c}(T)$ contains a FAI $A \Rightarrow C$ with $C = T(A \Rightarrow B) \otimes B$. For a FAI $A \Rightarrow B$ and a fuzzy set $T$ of FAIs, we define a *degree $|A \Rightarrow B|_T^\omega \in L$ to which $A \Rightarrow B$ is provable from $T$* by

$$|A \Rightarrow B|_T^\omega = \bigvee\{c \in L \mid \mathrm{c}(T) \vdash_\omega A \Rightarrow c \otimes B\}. \tag{12}$$

We now have a graded-style completeness for fuzzy attribute logic over arbitrary complete residuated lattices saying that a degree of provability equals a degree of semantic entailment:

THEOREM 3.7 *For each fuzzy set $T$ of FAIs and each $A \Rightarrow B$, we have $|A \Rightarrow B|_T^\omega = ||A \Rightarrow B||_T$.*

*Proof* The claim follows from Theorem 3.6 using the fact that $||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_{\mathrm{c}(T)} = 1\}$, see (Bělohlávek and Vychodil, 2005c). □

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