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Fuzzy Concept Lattices Constrained by Hedges

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We study concept lattices constrained by hedges. The principal aim is to control, in a parameterical way, the size of concept lattices, i.e. the number of conceptual clusters extracted from data. The paper presents theoretical insight, comments, and examples. We introduce new, parameterized, concept-forming operators and study their properties. We obtain an axiomatic characterization of the concept-forming operators. Then, we show that a concept lattice with hedges is indeed a complete lattice which is isomorphic to an ordinary concept lattice. We describe the isomorphism and its inverse. These mappings serve as translation procedures. As a consequence, we obtain a theorem characterizing the structure of concept lattices with hedges which generalizes the well-known main theorem of ordinary concept lattices. Furthermore, the isomorphism and its inverse enable us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. Further insight is provided for boundary choices of hedges. We demonstrate by experiments that the size reduction using hedges as parameters is smooth.

Keywords: formal concept analysis, concept lattice, fuzzy logic, fuzzy attribute, hedge

1. Problem Setting

Data tables describing objects and their attributes represent perhaps the most common form of data. Among several methods for analysis of object-attribute data, formal concept analysis (FCA) is becoming increasingly popular, see [13, 14]. The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data along with a partial order of these clusters (called conceptual hierarchy). Formal concepts correspond to maximal rectangles in a data table and are easily interpretable by users. FCA is basically being used two ways. First, as a direct method of data analysis in which case the hierarchically ordered collection of formal concepts extracted from data is presented to a user/expert for further analysis, see e.g. [13] for such examples of FCA applications. Second, as a data preprocessing method in which case the extracted clusters are used for further processing, for instance, for mining non-redundant sets of association rules. As with other methods of exploratory data analysis, the number of formal concepts extracted from data can be large. Since a large collection of formal concepts is not directly comprehensible by a user, methods are needed to help solve this problem.

In this paper, we propose a method to control, in a parameterical way, the number of formal concepts extracted from data tables with fuzzy attributes. The parameters we use are particular linguistic hedges, see e.g. [19, 27]. The paper is an extension of our conference paper [8]. The main idea of our approach consists in a modification of concept-forming operators by means of linguistic hedges. An important feature of our approach is that the verbal description and hence the meaning of formal concepts, i.e. of clusters extracted from data, does not change. That is, the formal concepts are still easily interpretable for users. Another important feature is that our approach is not an ad-hoc modification of the original method. Our method retains its theoretical and computational tractability, with the same order of complexity as with the original method. The original method can be seen as a particular case of our new method with the parameters (hedges) being identity mappings. Stronger hedges lead to smaller numbers of extracted formal concepts – this is the basic purpose of using hedges as parameters in our method.

The paper is organized as follows. Section 2 surveys preliminaries from fuzzy logic and fuzzy sets, and from formal concept analysis. In Section 3, we present our method and perform its theoretical analysis, mainly an analysis of issues related to applications of formal concept analysis. Section 4 contains examples and experiments demonstrating the reduction of the number of extracted formal concepts.

2. Preliminaries

2.1. Fuzzy Sets and Fuzzy Relations

Fuzzy logic and fuzzy set theory are formal frameworks for a manipulation of a particular form of imperfection called fuzziness (vagueness). Contrary to classical logic, fuzzy logic uses a scale L of truth degrees, a most common choice being $L = [0,1]$ (real unit interval) or some subchain of $[0,1]$. This enables to consider intermediate truth degrees of propositions, e.g. "object x has attribute y" has a truth degree 0.8 indicating that the proposition is almost true. In addition to a set L of truth degrees, one has to pick an appropriate collection of logical connectives (implication, conjunction, …). A general choice of a set of truth degrees plus logical connectives is represented by so-called complete residuated lattices (equipped possibly with additional operations). The rest of this section presents an introduction to fuzzy logic notions we need in the sequel. Details can be found e.g. in [3, 16, 17], a good introduction to fuzzy logic and fuzzy sets is presented in [19].

A complete residuated lattice [17] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \qquad (1)$$

for each $a, b, c \in L$. We will use the concept of a truth-stressing hedge [17, 18] which is a particular case of a general concept of a linguistic hedge proposed by Zadeh [19, 27]. By a truth-stressing hedge (shortly, a hedge) on $\mathbf{L}$ we mean a unary mapping * on L satisfying

$$1^* = 1 \qquad (2)$$

$$a^* \leq a \qquad (3)$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^* \qquad (4)$$

$$a^{**} = a^* \qquad (5)$$

for each $a, b \in L$. Elements a of L are called truth degrees. Operations $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [17, 18]. Properties (3)-(5) have natural interpretations, e.g. Eq. (3) can be read: "if a is very true, then a is true", Eq. (4) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

Łukasiewicz:
$$a \otimes b = \max(a + b - 1, 0), \quad a \rightarrow b = \min(1 - a + b, 1), \qquad (6)$$

Gödel:
$$a \otimes b = \min(a, b), \quad a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \qquad (7)$$

Goguen (product):
$$a \otimes b = a \cdot b, \quad a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \qquad (8)$$

In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0,1]$ to L.

Two boundary cases of (truth-stressing) hedges are

(i) identity, i.e. $a^* = a$ ($a \in L$);

(ii) globalization [26]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (9)$$

A special case of a complete residuated lattice with hedge is a two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$.

Having $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "a degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{^{a_1}/u_1, \ldots, ^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. For brevity, we introduce the following convention: we write $\{\ldots, u, \ldots\}$ instead of $\{\ldots, ^1/u, \ldots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u, ^{0.5}/v\}$ instead of $\{^1/u, ^{0.5}/v, ^0/w\}$, etc.

Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between U and V can be thought of as $\mathbf{L}$-sets in the universe $U \times V$. That is, a binary $\mathbf{L}$-relation $R \in \mathbf{L}^{U \times V}$ between a set U and a set V is a mapping assigning to each $u \in U$ and each $v \in V$ a truth degree $R(u, v) \in L$ (a degree to which u and v are related by R). An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called crisp if $A(u) \in \{0,1\}$ for each $u \in U$. Crisp $\mathbf{L}$-sets can be identified with (characteristic functions of) ordinary sets: crisp $\mathbf{L}$-set $A \in \mathbf{L}^U$ corresponds to the ordinary set $\{u \in U \,|\, A(u) = 1\}$. Therefore, for a crisp A, we also write $u \in A$ for $A(u) = 1$ and $u \notin A$ for $A(u) = 0$. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called empty (denoted by $\emptyset$) if $A(u) = 0$ for each $u \in U$; $A \in \mathbf{L}^U$ is called full (denoted by U) if $A(u) = 1$ for each $u \in U$. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called a singleton if there is $u \in U$ such that $A(v) = 0$ for each $v \neq u$. In such a case, we denote A by $\{^a/u\}$ where $a = A(u)$. Note that we allow $A(u) = 0$, i.e. a singleton may be an empty fuzzy set.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A,B) = \bigwedge_{u\in U}\big(A(u) \to B(u)\big) \quad \ldots\ldots \quad (10)$$

of A in B generalizing the classical subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [3, 17]. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice with a hedge.

In the sequel we will take advantage of one the common methods of representing $\mathbf{L}$-sets (fuzzy sets) by $\mathbf{2}$-sets (ordinary sets) [3]: for $A \in \mathbf{L}^U$ we define $\lfloor A \rfloor \in \mathbf{2}^{U\times L}$ by

$$\lfloor A \rfloor = \{\langle u,a\rangle \in U \times L \,|\, a \leq A(u)\}. \quad \ldots\ldots \quad (11)$$

Described verbally, $\lfloor A \rfloor$ can be considered as an area under the membership function $A\colon U \to L$. For $B \in \mathbf{2}^{U\times L}$ we define $\lceil B \rceil \in \mathbf{L}^U$ by

$$\lceil B \rceil(u) = \bigvee\{a \in L \,|\, \langle u,a\rangle \in B\} \quad \ldots\ldots \quad (12)$$

for each $u \in U$.

2.2. Formal Concept Analysis of Data with Fuzzy Attributes

In its basic setting, FCA can be applied to data with bivalent (crisp) attributes. We are interested in an extension of FCA which can be applied to data with fuzzy attributes. In fact, several such extensions have been proposed, see e.g. [9] for an overview. We are interested in an approach presented e.g. in [3, 24] since it is the most elaborated one. In what follows, we present basic notions.

A data table with fuzzy attributes, which is the input to FCA of data with fuzzy attributes, can be represented by a triplet $\langle X,Y,I\rangle$ where X is a finite set of objects, Y is a finite set of attributes, and $I \in \mathbf{L}^{X\times Y}$ is a binary fuzzy relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x,y) \in L$ to which x has y. $\langle X,Y,I\rangle$ can be thought of as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x,y)$, see e.g. **Fig. 1** in Section 4.

For $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^{\Uparrow} \in \mathbf{L}^Y$ (fuzzy set of attributes) and $B^{\Downarrow} \in \mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\Uparrow}(y) = \bigwedge_{x\in X}\big(A(x) \to I(x,y)\big) \quad \ldots\ldots \quad (13)$$

$$B^{\Downarrow}(x) = \bigwedge_{y\in Y}\big(B(y) \to I(x,y)\big). \quad \ldots\ldots \quad (14)$$

Described verbally, $A^{\Uparrow}$ is the fuzzy set of all attributes from Y shared by all objects from A (and similarly for $B^{\Downarrow}$). A formal (fuzzy) concept of $\langle X,Y,I\rangle$ is any pair $\langle A,B\rangle$ of $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ satisfying $A^{\Uparrow} = B$ and $B^{\Downarrow} = A$. That is, a formal concept consists of a fuzzy set A (so-called extent) of objects which fall under the concept and a fuzzy set B (so-called intent) of attributes which fall under the concept such that A is the fuzzy set of all objects from X sharing all attributes from B and, conversely, B is the fuzzy set of all attributes from Y shared by all objects from A. Formal concepts represent conceptual clusters hidden in the data table $\langle X,Y,I\rangle$. The notion of a formal concept is inspired by a traditional understanding of human concepts which goes back to Port-Royal logic.

A collection $\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \,|\, A^{\Uparrow} = B, B^{\Downarrow} = A\}$, i.e. a collection of all conceptual clusters of $\langle X,Y,I\rangle$, can be equipped with a partial order $\leq$ modeling the subconcept-superconcept hierarchy (e.g., *dog* $\leq$ *mammal*) defined by

$$\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle \text{ iff } A_1 \subseteq A_2 \ (\text{iff } B_2 \subseteq B_1). \quad (15)$$

Note that $^{\Uparrow}$ and $^{\Downarrow}$ form a so-called fuzzy Galois connection [3] and that $\mathcal{B}(X,Y,I)$ is in fact a set of all fixed points of $^{\Uparrow}$ and $^{\Downarrow}$. Under $\leq$, $\mathcal{B}(X,Y,I)$ happens to be a complete lattice, called a fuzzy concept lattice of $\langle X,Y,I\rangle$. The basic structure of fuzzy concept lattices is described by the so-called main theorem of concept lattices [3, 4], the first part of which is given by the following theorem.

Theorem 1—see [4]: The set $\mathcal{B}(X,Y,I)$ is under $\leq$ a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle,$$
$$\bigvee_{j\in J}\langle A_j,B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\Uparrow\Downarrow}, \bigcap_{j\in J} B_j\rangle.$$

□

For a detailed information on formal concept analysis of data tables with fuzzy attributes we refer to [3, 8]. Formal concept analysis of data tables with binary attributes is thoroughly studied in [13, 14] where a reader can find theoretical foundations, methods and algorithms, and applications in various areas.

3. Concept Lattices with Hedges

3.1. Definition and Remarks

We suppose that we are given a complete residuated lattice $\mathbf{L}$, and two hedges, $*_X$ and $*_Y$ on $\mathbf{L}$. Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \to L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has attribute y. The triplet $\langle X,Y,I\rangle$ represents a data table with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x,y)$.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x)^{*_X} \to I(x,y)) \quad \ldots\ldots \quad (16)$$

and

$$B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y)^{*_Y} \to I(x,y)). \quad \ldots\ldots \quad (17)$$

Using basic rules of predicate fuzzy logic, $A^{\uparrow}(y)$ is the truth degree of "for each $x \in X$: if it is very true that x belongs from A then x has y" where "very true" is interpreted by $*_X$. Similarly for $B^{\downarrow}$ with "very true" interpreted by $*_Y$. That is, $A^{\uparrow}$ is a fuzzy set of attributes common to all objects for which it is very true that they belong to A, and $B^{\downarrow}$ is a fuzzy set of objects sharing all attributes for which it is very true that they belong to B. The set

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ thus contains all pairs $\langle A, B\rangle$ such that A is the collection of all objects that have all the attributes of "very B", and B is the collection of all attributes that are shared by all the objects of "very A". For the sake of brevity, we use also $\mathcal{B}(X^*, Y^*, I)$ instead of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Also, we omit $*$ if it is the identity and write e.g. only $\mathcal{B}(X, Y^{*_Y}, I)$. Given $*_X$ and $*_Y$ as parameters, elements $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively; $\mathcal{B}(X^*, Y^*, I)$ will be called a concept lattice of $\langle X, Y, I\rangle$. Both the extent A and the intent B are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not necessarily 0 and 1.

For $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1).$$

This defines a subconcept-superconcept hierarchy on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Note that we have $A^{\uparrow} = (A^{*_X})^{\Uparrow}$ and $B^{\downarrow} = (B^{*_Y})^{\Downarrow}$, see Eqs. (13) and (14).

Example 1: (1) Let both $*_X$ and $*_Y$ be identities on L. Then $\mathcal{B}(X, Y, I)$, i.e. $\mathcal{B}(X^*, Y^*, I)$, is what is called a (fuzzy) concept lattice, see e.g. [3, 24]. Axiomaticcharacterization of mappings $^{\uparrow}$ and $^{\downarrow}$ is given in [1].

(2) Recall from [7] that a crisply generated formal concept of $\langle X, Y, I\rangle$ is a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ ($*_X$ and $*_Y$ are identities) which is generated by a crisp (fuzzy) set of attributes, i.e. there is $D \in \{0, 1\}^Y$ such that $A = D^{\Downarrow}$ and $B = A^{\Uparrow}$. Crisply generated formal concepts may be thought of as the important ones. The number of crisply generated concepts is considerably smaller than the number of all formal concepts, see [7]. Now, it can be easily shown that if $*_X$ is the identity and $*_Y$ is the globalization on L, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is just the set of all crisply generated concepts.

(3) It can be shown that what is called a fuzzy concept lattice in [11] is in fact a structure isomorphic to $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ with $*_X$ and $*_Y$ being identity and globalization, respectively. If, on the other hand, $*_X$ and $*_Y$ are globalization and identity, respectively, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to what is called a one-sided fuzzy concept lattice in [20].

(4) An attribute implication [10] is an expression $A \Rightarrow B$ where $A, B \in L^Y$ are fuzzy sets of attributes. The degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = \textstyle\bigwedge_{x\in X} S(A, I_x)^* \to S(B, I_x).$$

Here, $I_x \in L^Y$ is a fuzzy set of attributes of object x, i.e. $I_x(y) = I(x, y)$, and $*$ is globalization on L. Then, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is the truth degree of "each object from X having all attributes from A has also all attributes from B". It can be shown that a set T of attribute implications is a base, i.e. T semantically entails exactly the set of all attribute implications which are fully true (i.e., to degree 1) in $\langle X, Y, I\rangle$, if and only if the set of all models of T (a fuzzy set of attributes in which all implications of T are true) equals the set of all intents of formal concepts from $\mathcal{B}(X^*, Y, I)$, see [10] for details.

3.2. The Structure of Concept Lattices with Hedges

A concept lattice (without hedges, i.e. with both $*_X$ and $*_Y$ being identity) is a complete lattice with infima and suprema corresponding to conceptual specifications and generalizations. Moreover, a characterization of concept lattices up to an isomorphism is known (see [14] for crisp case and [3] for fuzzy setting). The question we are going to answer is: What is the structure of concept lattices with hedges, i.e. the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$? The answer is not obvious. For instance, neither of the composed mappings $^{\uparrow\downarrow}$ and $^{\downarrow\uparrow}$ is a closure operator. Indeed, neither $A \subseteq A^{\uparrow\downarrow}$ nor $B \subseteq B^{\downarrow\uparrow}$ is true in general [6]. In order to answer our question, we proceed as follows: First, we find an ordinary Galois connection $\langle{}^{\curlywedge}, {}^{\curlyvee}\rangle$ between sets such that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to the the lattice of fixpoints of $\langle{}^{\curlywedge}, {}^{\curlyvee}\rangle$. In addition to that, we describe the isomorphism and its inverse. Second, since $\langle{}^{\curlywedge}, {}^{\curlyvee}\rangle$ is a Galois connection between sets, the lattice of its fixpoint obeys the so-called main theorem of concept lattices. Applying the isomorphism and its inverse, we get the theorem describing the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Denote

$$*_X(L) = \{a^{*_X} \mid a \in L\} \text{ and } *_Y(L) = \{a^{*_Y} \mid a \in L\}.$$

Furthermore, for $A \in L^U$, $A' \subseteq U \times L$, and $* : L \to L$, define $A^* \in L^U$ and $A'^* \subseteq U \times L$ by $A^*(u) = (A(u))^*$ and $A'^* = \{\langle x, a^*\rangle \mid \langle x, a\rangle \in A'\}$.

Lemma 2: For $A \subseteq X \times *_X(L)$ we have $A \subseteq \lfloor \lceil A\rceil^{*_X} \rfloor^{*_X}$. If $B = \lfloor B'\rfloor$ for some $B' \in L^Y$ then $\lceil B^{*_Y}\rceil = \lceil B\rceil^{*_Y}$.

Proof. Directly from definitions. □

Define mappings $^{\curlywedge} : X \times *_X(L) \to Y \times *_Y(L)$ and $^{\curlyvee} : Y \times *_Y(L) \to X \times *_X(L)$ by

$$A^{\curlywedge} = \lfloor \lceil A\rceil^{\uparrow} \rfloor^{*_Y} \text{ and } B^{\curlyvee} = \lfloor \lceil B\rceil^{\downarrow} \rfloor^{*_X}. \quad \ldots\ldots (18)$$

Lemma 3: The pair $\langle{}^{\curlywedge}, {}^{\curlyvee}\rangle$ forms a Galois connection between sets $X \times *_X(L)$ and $Y \times *_Y(L)$.

Proof. Antitony: $A_1 \subseteq A_2$ implies $\lceil A_1\rceil \subseteq \lceil A_2\rceil$ which implies $\lceil A_2\rceil^{\uparrow} \subseteq \lceil A_1\rceil^{\uparrow}$ which implies $\lfloor \lceil A_2\rceil^{\uparrow} \rfloor \subseteq \lfloor \lceil A_1\rceil^{\uparrow} \rfloor$ which implies $A_2^{\curlywedge} = \lfloor \lceil A_2\rceil^{\uparrow} \rfloor^{*_X} \subseteq \lfloor \lceil A_1\rceil^{\uparrow} \rfloor^{*_X} = A_1^{\curlywedge}$. Dually, $B_1 \subseteq B_2$ implies $B_2^{\curlyvee} \subseteq B_1^{\curlyvee}$.

Extensivity: Using Lemma 2, $A^{\curlywedge\curlyvee} = \lfloor \lceil \lfloor \lceil A\rceil^{\uparrow} \rfloor^{*_Y} \rceil^{\downarrow} \rfloor^{*_X} = \lfloor \lceil \lfloor \lceil A\rceil^{\uparrow} \rfloor \rceil^{*_Y\downarrow} \rfloor^{*_X} = \lfloor \lceil A\rceil^{\uparrow *_Y \downarrow} \rfloor^{*_X} = \lfloor \lceil A\rceil^{\uparrow\downarrow} \rfloor^{*_X} \supseteq \lfloor \lceil A\rceil^{*_X} \rfloor^{*_X} \supseteq A$. Dually, $B \subseteq B^{\curlyvee\curlywedge}$. □

It is well-known (see e.g. [14]) that each Galois connection $\langle{}^{\curlywedge}, {}^{\curlyvee}\rangle$ between sets U and V is induced by some binary relation $I_{\langle\curlywedge,\curlyvee\rangle} \subseteq U \times V$. Namely, $I_{\langle\curlywedge,\curlyvee\rangle}$ is given by $\langle u,v\rangle \in I_{\langle\curlywedge,\curlyvee\rangle}$ iff $v \in \{u\}^{\curlywedge}$. Then we have $A^{\curlywedge} = \{v \in V \mid$ for each $u \in A : \langle u,v\rangle \in I_{\langle\curlywedge,\curlyvee\rangle}\}$ for any $A \subseteq U$, and $B^{\curlyvee} = \{u \in U \mid$ for each $v \in B : \langle u,v\rangle \in I_{\langle\curlywedge,\curlyvee\rangle}\}$ for any $B \subseteq V$. Furthermore, in such a case, the set $\mathcal{B}(U,V,\langle^{\curlywedge},^{\curlyvee}\rangle) = \{\langle A,B\rangle \in 2^U \times 2^V \mid A^{\curlywedge} = B, B^{\curlyvee} = A\}$ of all fixpoints of $\langle^{\curlywedge},^{\curlyvee}\rangle$ (which is in fact the ordinary concept lattice $\mathcal{B}(U,V,I_{\langle\curlywedge,\curlyvee\rangle})$) obeys the so-called main theorem of concept lattices:

Theorem 4—[14]: (1) $\mathcal{B}\left(U,V,I_{\langle\curlywedge,\curlyvee\rangle}\right)$ is under $\leq$, defined by $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$, a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\curlyvee\curlywedge}\rangle ,$$

$$\bigvee_{j\in J}\langle A_j,B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\curlywedge\curlyvee}, \bigcap_{j\in J} B_j\rangle .$$

(2) Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K,\leq\rangle$ is isomorphic to $\mathcal{B}\left(U,V,I_{\langle\curlywedge,\curlyvee\rangle}\right)$ iff there are mappings $\gamma : U \to K$, $\mu : V \to K$ such that

(i) $\gamma(U)$ is $\bigvee$-dense in K, $\mu(V)$ is $\bigwedge$-dense in V;

(ii) $\gamma(u) \leq \mu(v)$ iff $\langle u,v\rangle \in I_{\langle\curlywedge,\curlyvee\rangle}$.

Lemma 5: The (ordinary) relation $I^{\times} = I_{\langle\curlywedge,\curlyvee\rangle}$ between $X \times *_X(L)$ and $Y \times *_Y(L)$ corresponding to a Galois connection $\langle^{\curlywedge},^{\curlyvee}\rangle$ defined by (18) is given by

$$\langle\langle x,a\rangle,\langle y,b\rangle\rangle \in I^{\times} \text{ iff } a \otimes b \leq I(x,y). \quad (19)$$

Proof. By the above remark, we have $\langle\langle x,a\rangle,\langle y,b\rangle\rangle \in I^{\times}$ iff $\langle y,b\rangle \in \{\langle x,a\rangle\}^{\curlywedge}$. By definition of $^{\curlywedge}$, this is equivalent to $\langle y,b\rangle \in \lfloor\lceil\{\langle x,a\rangle\}\rceil^{\uparrow}\rfloor^{*_Y}$. Since $\lfloor\lceil\{\langle x,a\rangle\}\rceil^{\uparrow}\rfloor^{*_Y} = \lfloor\{{}^{a}\!/x\}^{\uparrow}\rfloor^{*_Y}$ and since the greatest c such that $\langle y,c\rangle \in \lfloor\{{}^{a}\!/x\}^{\uparrow}\rfloor^{*_Y}$ is $c = (\{{}^{a}\!/x\}^{\uparrow}(y))^{*_Y}$, the last assertion is equivalent to $b \leq (\{{}^{a}\!/x\}^{\uparrow}(y))^{*_Y}$. Since $b = b^{*_Y}$, this is equivalent to $b \leq \{{}^{a}\!/x\}^{\uparrow}(y)$. Now, $\{{}^{a}\!/x\}^{\uparrow}(y) = a^{*_X} \to I(x,y) = a \to I(x,y)$, whence $b \leq \{{}^{a}\!/x\}^{\uparrow}(y)$ is equivalent to $a \otimes b \leq I(x,y)$ by adjointness. □

In the rest, $I^{\times}$ always denotes the relation from Lemma 5.

Theorem 6: Every concept lattice with hedges $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is isomorphic to the ordinary concept lattice $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$. The isomorphism $h : \mathcal{B}(X^{*_X},Y^{*_Y},I) \to \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$ and its inverse $g : \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times}) \to \mathcal{B}(X^{*_X},Y^{*_Y},I)$ are given by

$$h(\langle A,B\rangle) = \langle \lfloor A\rfloor^{*_X}, \lfloor B\rfloor^{*_Y}\rangle \quad (20)$$

$$g(\langle A',B'\rangle) = \langle \lceil A'\rceil^{\uparrow\downarrow}, \lceil B'\rceil^{\downarrow\uparrow}\rangle. \quad (21)$$

Proof. The theorem can be proven by showing that (a) h and g are defined correctly, (b) h is order-preserving, and (c) $g(h(\langle A,B\rangle)) = \langle A,B\rangle$ and $h(g(\langle A',B'\rangle)) = \langle A',B'\rangle$. We give only a sketch.

"(a)": We need to show that for each $\langle A,B\rangle$ from $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ and each $\langle A',B'\rangle$ from $\in \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$ we have $g(A',B') \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$ and $h(A',B') \in \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$. This can be verified using previous propositions. "(b)" is evident. "(c)": Can be verified using previous propositions. □

The following is our main theorem describing the structure of concept lattices with hedges.

Theorem 7—main theorem for concept lattices with hedges: (1) $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is under $\leq$ a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle (\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}\rangle . \quad (22)$$

$$\bigvee_{j\in J}\langle A_j,B_j\rangle = \langle (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle . \quad (23)$$

(2) Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K,\leq\rangle$ is isomorphic to $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ iff there are mappings $\gamma : X \times *_X(L) \to K$, $\mu : Y \times *_Y(L) \to K$ such that

(i) $\gamma(X \times *_X(L))$ is $\bigvee$-dense in K, $\mu(Y \times *_Y(L))$ is $\bigwedge$-dense in V;

(ii) $\gamma(x,a) \leq \mu(y,b)$ iff $a \otimes b \leq I(x,y)$.

Proof. Use Theorem 6 and apply Theorem 4 to $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$. Then, using h and g, translate the theorem characterizing $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$ to a theorem characterizing $\mathcal{B}(X^{*_X},Y^{*_Y},I)$. Doing that, we obtain our theorem with formulas for $\bigwedge_{j\in J}\langle A_j,B_j\rangle$ and $\bigvee_{j\in J}\langle A_j,B_j\rangle$ which are cumbersome. However, they can be simplified to Eqs. (22) and (23). □

If $*_X = \mathrm{id}$, a hedge is applied only to attributes. In such a case, we denote the concept lattice with hedges by $\mathcal{B}(X,Y^{*_Y},I)$. This is an important special case. If $*_Y$ is globalization (first boundary possibility), $\mathcal{B}(X,Y^{*_Y},I)$ is just the lattice of crisply generated concepts [7]. If $*_Y$ is identity (second boundary possibility), $\mathcal{B}(X,Y^{*_Y},I)$ is the whole fuzzy concept lattice [4]. In general, $*_Y$ (possibly between globalization and identity) controls the meaning of "having all attributes from (the intent) B". Loosely speaking, paying attention to $\langle A,B\rangle \in \mathcal{B}(X,Y^{*_Y},I)$ means that we do not put any restriction on extents A (any closed fuzzy set of objects of objects is good), but use $*_Y$ to impose a restriction on intents B. This follows intuition. For instance, an intent B which contains each attribute to degree 0.5 might seem not natural. If this is our view, we can take globalization for $*_Y$ and the corresponding concept $\langle A,B\rangle$ disappears (does not belong to $\mathcal{B}(X,Y^{*_Y},I)$).

We are going to show that if $*_X = \mathrm{id}$, one can answer several important questions. The first theorem shows that concepts from $\mathcal{B}(X,Y^{*_Y},I)$ are particular concepts from the whole $\mathcal{B}(X,Y,I)$.

Theorem 8: $\mathcal{B}(X,Y^{*_Y},I) \subseteq \mathcal{B}(X,Y,I)$. Moreover, $\mathcal{B}(X,Y^{*_Y},I) = \{\langle A,B\rangle \in \mathcal{B}(X,Y,I) \mid A = D^{\downarrow}$ for some $D \in *_Y(L)^Y\}$.

Proof. "$\subseteq$": If $\langle A,B\rangle \in \mathcal{B}(X,Y^{*_Y},I)$ then clearly, $D := B^{*_Y} \in *_Y(L)^Y$ and $A = D^{\downarrow}$. Furthermore, $A^{\Uparrow} = A^{\uparrow} = B$ and $A = B^{\Downarrow} = B^{*_Y\Downarrow} \supseteq B^{\Downarrow} = A^{\Uparrow\Downarrow} \supseteq A$, whence $B^{\Downarrow} = A$. That is, $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$.

"$\supseteq$": Let $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ such that $A = D^{\downarrow}$ for some $D \in *_Y(L)^Y$. We need to show $\langle A,B\rangle \in \mathcal{B}(X,Y^{*_Y},I)$ for which it is sufficient to see $A = B^{\downarrow}$. We have $A = D^{\downarrow} = D^{*_Y\Downarrow} = D^{*_Y\Downarrow\Uparrow*_Y\Downarrow} = D^{\Downarrow\Uparrow*_Y\Downarrow} = B^{*_Y\Downarrow} = B^{\downarrow}$. □

In the general case, $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ need not be a subset of $\mathcal{B}(X,Y,I)$ as shown by the following example.

Example 2: Take a Łukasiewicz structure on $[0,1]$, let both $*_X$ and $*_Y$ be globalizations, and consider the following data table

I	y_1	y_2
x_1	1	0.5
x_2	0.7	0.1

One can check that for $A = \{ {}^{1}/x_1, {}^{0.7}/x_2 \}$ and $B = \{ {}^{1}/y_1, {}^{0.5}/y_2 \}$, $\langle A,B\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$ but $\langle A,B\rangle \notin \mathcal{B}(X,Y,I)$.

$*(L)$ is in fact the set of all fixpoints of $*$, i.e. those $a \in L$ for which $a^* = a$. The next theorem shows that the smaller the set of fixpoints of $*_Y$, the larger the reduction.

Theorem 9: If $*_1(L) \subseteq *_2(L)$ then $\mathcal{B}(X,Y^{*_1},I) \subseteq \mathcal{B}(X,Y^{*_2},I)$.

Proof. Follows immediately from Theorem 8. □

Theorem 10: If $*_X = \mathrm{id}$, formula (22) simplifies to any of the following forms:

$$\textstyle\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle \quad . \quad . \quad . \quad (24)$$

$$\textstyle\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle. \quad . \quad . \quad . \quad (25)$$

Proof. First, we show that in this case, $(\cap_j A_j)^{\uparrow\downarrow} = \cap_j A_j$: On the one hand, $\cap_j A_j \subseteq (\cap_j A_j)^{\Uparrow\Downarrow} = (\cap_j A_j)^{\uparrow\Downarrow} \subseteq (\cap_j A_j)^{\uparrow\downarrow}$. On the other hand, $(\cap_j A_j)^{\uparrow\downarrow} \subseteq \cap_j A_j$ iff for each $j \in J$ we have $(\cap_j A_j)^{\uparrow\downarrow} \subseteq A_j$ which is true. Indeed, $(\cap_j A_j) \subseteq A_j$ implies $(\cap_j A_j)^{\uparrow\downarrow} \subseteq A_j^{\uparrow\downarrow}$ and $A_j^{\uparrow\downarrow} = A_j$ since A_j is an extent.

Second, since

$$\langle(\cap_j A_j),(\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle = \langle(\cap_j A_j)^{\uparrow\downarrow},(\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle \in \mathcal{B}(X,Y^{*_Y},I),$$

Theorem 8 yields

$$\langle(\cap_j A_j),(\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle \in \mathcal{B}(X,Y,I).$$

Now, observe that the intent corresponding to $\cap_j A_j$ in $\mathcal{B}(X,Y,I)$ is $(\cup_j B_j)^{\Downarrow\Uparrow}$ (see e.g. [3, 4]). This yields $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} = (\cup_j B_j)^{\Downarrow\Uparrow}$. Furthermore, $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} \subseteq (\cup_j B_j)^{\downarrow\uparrow} \subseteq (\cup_j B_j)^{\Downarrow\Uparrow} = (\cup_j B_j^{*_Y})^{\downarrow\uparrow}$, whence $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} = (\cup_j B_j)^{\downarrow\uparrow}$. The proof is finished. □

As a corollary, we get the following assertion.

Theorem 11: If $*_X = \mathrm{id}$, then $\mathcal{B}(X,Y^{*_Y},I)$ is a $\bigwedge$-sublattice of $\mathcal{B}(X,Y,I)$.

Proof. Follows from Theorem 10 and the fact that the infimum of $\langle A_j,B_j\rangle$'s in $\mathcal{B}(X,Y,I)$ is given by Eq. (25), see [3]. □

The following example shows that $\mathcal{B}(X,Y^{*_Y},I)$ need not be a $\bigvee$-sublattice of $\mathcal{B}(X,Y,I)$.

Example 3: Take a Łukasiewicz structure on $[0,1]$ (but this works for Gödel and product as well), let $*_Y$ be globalization, and consider the following data table

I	y_1	y_2	y_3
x_1	0.3	0.5	0.4
x_2	0.2	0.6	0.1

Then both $B_1 = \{ {}^{1}/y_1, {}^{1}/y_2 \}^{\downarrow\uparrow} = \{ {}^{1}/y_1, {}^{1}/y_2, {}^{0.9}/y_3 \}$ and $B_2 = \{ {}^{1}/y_2, {}^{1}/y_3 \}^{\downarrow\uparrow} = \{ {}^{0.9}/y_1, {}^{1}/y_2, {}^{1}/y_3 \}$ are intents in $\mathcal{B}(X,Y^{*_Y},I)$. However, since $B_1 \cap B_2 \neq (B_1 \cap B_2)^{\downarrow\uparrow}$, suprema in $\mathcal{B}(X,Y^{*_Y},I)$ and $\mathcal{B}(X,Y,I)$ are different.

The following theorem shows that if both $*_X$ and $*_Y$ are globalizations (boundary case, largest restriction), $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is in fact isomorphic to an ordinary concept lattice given by 1-cut ${}^{1}I = \{\langle x,y\rangle \mid I(x,y) = 1\}$ of I. Note that the data table $\langle X,Y,{}^{1}I\rangle$ results from $\langle X,Y,I\rangle$ by keeping entries with 1's and deleting (replacing by 0) all other entries.

Theorem 12: If $*_X$ and $*_Y$ are globalizations, $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is isomorphic to *(*ordinary*)* concept lattice $\mathcal{B}\left(X,Y,{}^{1}I\right)$.

Proof. By direct verification. □

3.3. Algorithms

A study of algorithms for constructing concept lattices with hedges is an important issue which we do not attempt to investigate in this paper. For the sake of completeness, we only mention that according to Theorem 6, one can proceed as follows: Transform the original table $\langle X,Y,I\rangle$ to $\langle X \times *_X(L), Y \times *_Y(L), I^{\times}\rangle$. Using algorithms for ordinary concept lattices, compute $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$. Using mapping g from Theorem 6, "translate" the ordinary concept lattice $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$ to $\mathcal{B}(X^{*_X},Y^{*_Y},I)$. Note that a direct algorithm for computing $\mathcal{B}(X,Y,I)$.

3.4. Refinement

There is a refinement of the approach we presented. We will use this refinement in Section 4. The refinement consists in considering families of hedges instead of one hedge $*_X$ for all objects and one hedge $*_Y$ for all attributes. Suppose that for each object $x \in X$ we are given a hedge $*_x$ on $\mathbf{L}$ and that for each attribute $y \in Y$ we are given a hedge $*_y$ on $\mathbf{L}$. For the sake of brevity, we will denote the collection of all $*_x$'s by $*_X$ and the collection of all $*_y$'s by $*_Y$. For fuzzy sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$, consider fuzzy sets $A^{\uparrow} \in \mathbf{L}^Y$ and $B^{\downarrow} \in \mathbf{L}^X$ defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}(A(x)^{*_x} \to I(x,y)) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}(B(y)^{*_y} \to I(x,y)). \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Clearly, Eqs. (16) and (17) are a particular case of Eqs. (26) and (27), respectively. Namely, the case when all $*_x$'s are the same, and all $*_y$'s are the same. Therefore, one can again denote by $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ the set

$$\mathcal{B}(X^{*_X},Y^{*_Y},I) = \{\langle A,B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$$

of all fixed points of the operators $^{\uparrow}$ and $^{\downarrow}$ defined by Eqs. (26) and (27). Our preliminary results indicate that main results we established in this section generalize to the refinement described here.

	birth rate		death rate	
	low	high	low	high
Brazil (B)	0	0.25	0.25	0.5
Czech Republic (C)	0.75	0	0	0.75
Eritrea (E)	0	1	0	0.75
France (F)	0.5	0	0	0.5
Germany (G)	1	0	0	0.75
Iran (I)	0	0.25	0.5	0.25
Israel (L)	0	0.5	0.25	0.5
Japan (J)	0.75	0	0	0.5
Kenya (K)	0	1	0	1
Malaysia (M)	0	0.75	0.5	0.25
Poland (P)	0.75	0	0	0.75
Russia (R)	0.75	0	0	1
Singapore (S)	0.75	0	0.75	0.25
United States (U)	0	0.25	0	0.5
Venezuela (V)	0	0.5	0.5	0.25

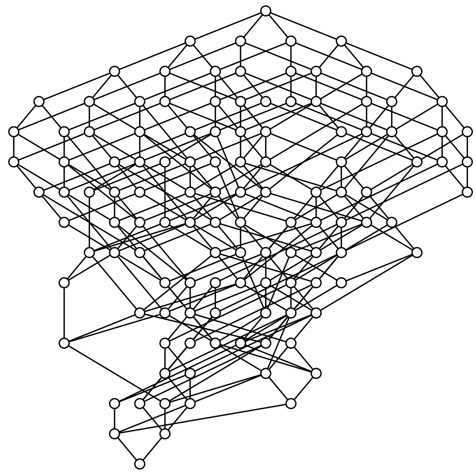

Fig. 1. Illustrative data table with fuzzy attributes and the corresponding fuzzy concept lattice (Hasse diagram of partially ordered formal concepts extracted from the table).

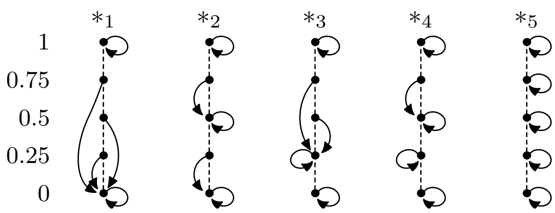

Fig. 2. Hedges on a five-element Łukasiewicz chain.

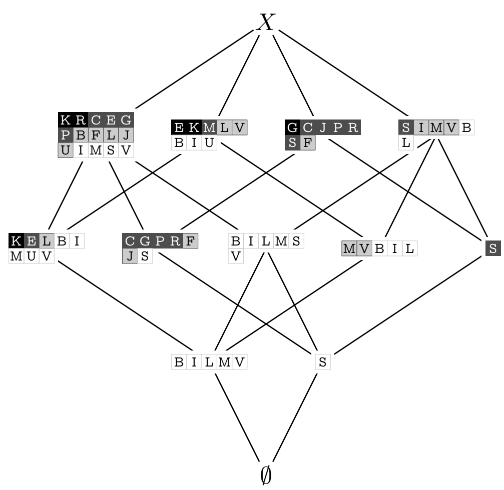

Fig. 3. Reduced fuzzy concept lattice.

4. Experiments and Examples

In this section we present examples which illustrate reduction of the size of fuzzy concept lattices by means of constraints imposed by hedges.

We take a five-element Łukasiewicz chain as our structure of truth degrees: We let $L = \{0, 0.25, 0.5, 0.75, 1\}$ be the set of truth degrees, $\wedge$ and $\vee$ being minimum and maximum, respectively. L, together with $\wedge$ and $\vee$, forms a five-element linearly ordered lattice with 0 and 1 being the least and the greatest element, respectively, with ordering of truth degrees given by $0 < 0.25 < 0.5 < 0.75 < 1$. The adjoint operations $\otimes$ and $\rightarrow$ are defined as follows: $a \otimes b = \max(a + b - 1, 0)$ and $a \rightarrow b = \min(1 - a + b, 1)$, cf. (6). Our structure of truth degrees is a finite subalgebra of the standard Łukasiewicz algebra defined on the real unit interval, see [3, 15, 17].

Consider now an illustrative data table with fuzzy attributes $\langle X, Y, I \rangle$ depicted in **Fig. 1 (left)**. The set X of objects consists of selected countries, the set Y of attributes consists of four attributes "low birth rate", "hight birth rate", "low death rate", and "hight death rate" describing birth/death rates in populations of the countries (the data was taken from the CIA Fact Book 2006 and then scaled to the truth degrees from L appropriately). Table entries indicate to which degree a given country has a low/high birth/death rate.

The fuzzy concept lattice $\mathcal{B}(X, Y, I)$ generated from this data table contains 121 fuzzy concepts (clusters). The hierarchy of fuzzy concepts is depicted in **Fig. 1 (right)**. As one can see, $\mathcal{B}(X, Y, I)$ is large and hardly graspable by users. Therefore, we can try to reduce the number of formal concepts in hierarchies by employing constraints by hedges. That is, instead of $\mathcal{B}(X, Y, I)$, we will now consider parameterized concept lattices $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ with hedges $*_X$ and $*_Y$ playing the role of parameters, see Section 3.4.

Our **L** admits five hedges. The hedges are depicted in **Fig. 2**. Arrows in **Fig. 2** indicate values a^{*_i} where $a \in L$ and $i = 1, \ldots, 5$. For instance, for $*_2 : L \rightarrow L$ we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, and $1^{*_2} = 1$. Analogously for the other hedges. Since $|X| = 15$ and $|Y| = 4$, there are $5^{15+4} = 19073486328125$ possibilities to select $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. For illustration, we take each $*_X$ to be identity (i.e., $a^{*_X} = a$ for each $a \in L$) and inspect the reduction of $\mathcal{B}(X, Y, I)$ depending on various choices of

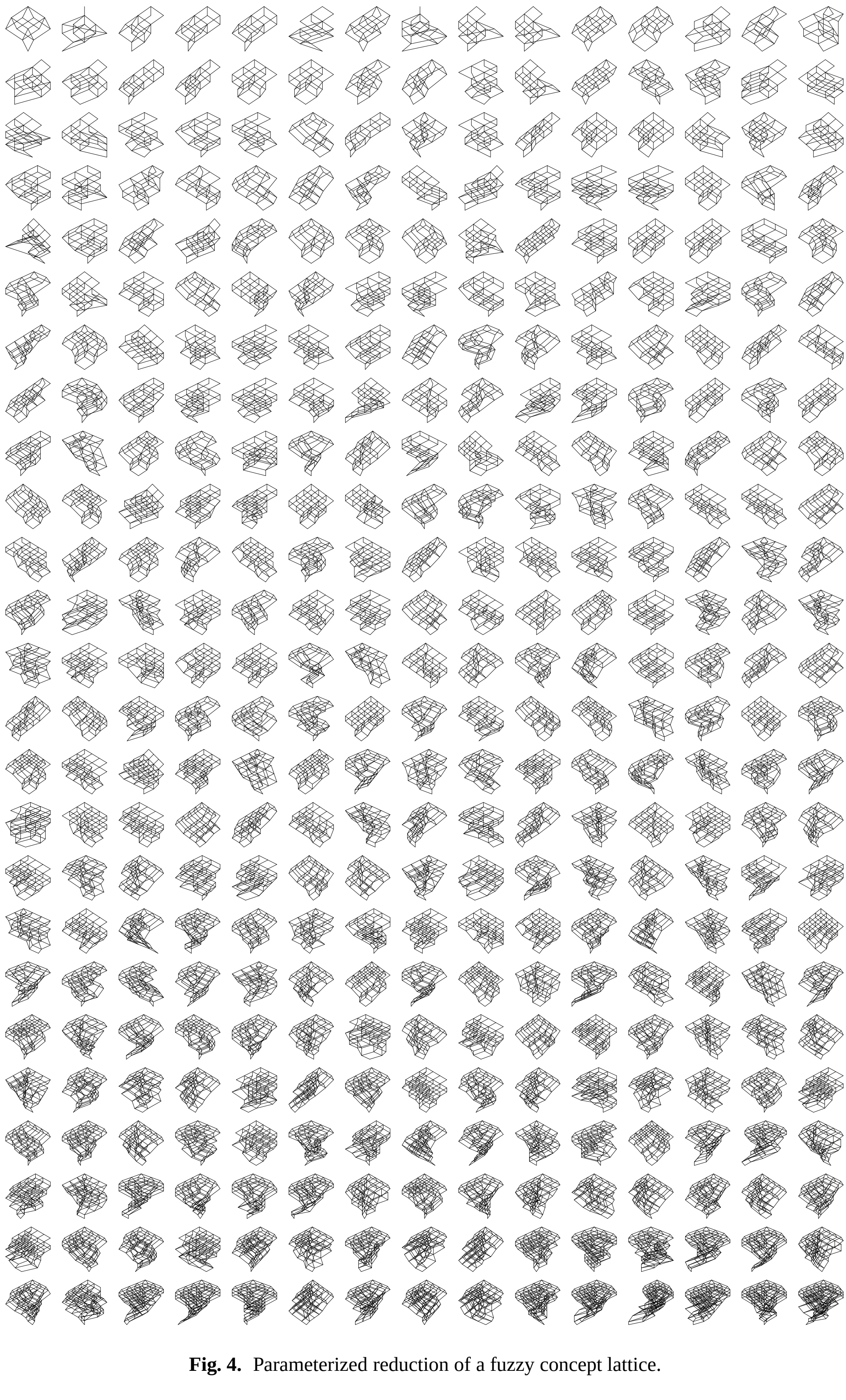

Fig. 4. Parameterized reduction of a fuzzy concept lattice.

$*_y$. Since $|Y| = 4$, there are $5^4 = 625$ possible constrained concept lattices lattices $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Due to Theorem 8 and its generalization for our refinement with different hedges for different attributes, each $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is a subset of the whole $\mathcal{B}(X,Y,I)$. Moreover, due to Theorem 9 and its generalization for our refinement with different hedges for different attributes, stronger hedges $*_y$ lead to stronger restrictions and thus to smaller concept lattices $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. If we compute concept lattices corresponding to all possible choices of $*_Y$'s, we arrive at 375 distinct sets of fuzzy concepts with sizes varying from 13 (least one) up to 121 (greatest one) concepts. Concept hierarchies corresponding to the distinct sets of fuzzy concepts are depicted in **Fig. 4**. Line diagrams in **Fig. 4** are sorted by the number of their edges. As we can see, there is a smooth transition from the least (most concise) hierarchy depicted in the top-left corner to the greatest (most detailed) hierarchy depicted in the bottom-right corner of the figure. The least fuzzy concept lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is depicted in more detail in **Fig. 3**.

In this figure we use the following method of describing formal concepts, i.e., conceptual clusters corresponding to the nodes of the diagram: each cluster is labeled by its extent (objects that fall under the concept), objects in extent are depicted by a color bar indicating degrees to which objects (represented by their abbreviations) fall under the concept (the darker the background color, the higher the degree; objects which belong to a concept to zero degree are not displayed).

The smoothness of transition from one hierarchy to another, which can be seen in **Fig. 4**, is a consequence of the fact that similar hedges yield similar structures of formal concepts. This important property can also be proved (we have estimation formulas saying to which degree the resulting concept lattices $\mathcal{B}(X, Y^{*_{Y1}}, I)$ and $\mathcal{B}(X, Y^{*_{Y2}}, I)$ are similar if $*_{y_1}$'s and $*_{y_2}$'s are going to be used). Let us note that some of the distinct sets of fuzzy concepts are isomorphic hierarchies (see the fourth and fifth diagrams in first line). Note that the choice of hedges $*_X$ and $*_Y$ is up to the user. Basically, the user needs not to define the hedges. Hedges are simple unary functions on the scale L of truth degrees and can easily be pre-computed automatically. The user's role is to say "use stronger hedges" if the resulting concept lattice is too large for the user's purpose, or to say "use weaker hedges" is the concept lattice is too small and the user wants to see more details.

5. Conclusion

The main motivation to study concept lattices with hedges is to control, in a parameterical way, the size of a concept lattice. Concept lattices with hedges generalize several previous approaches to formal concept analysis of data with fuzzy attributes. The paper presents theoretical insight to reducing the size of a fuzzy concept lattice using hedges. In particular, we showed a generalization of the main theorem of concept lattices. According to this, a concept lattice with hedges is indeed a complete lattice. Furthermore, it is isomorphic to an ordinary concept lattice, with a well-described isomorphism and its inverse which serve as translation procedures. Among other things, this enables us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. Further insight is provided in case one uses hedges only for attributes. Examples demonstrate that the size reduction using hedges as a parameter is smooth. Future research needs to focus on further theoretical insight (e.g., for case when both hedges are used simultaneously), on the refinement described in Section 3.4, and on combination of using hedges with other methods for reduction of the size of a fuzzy concept lattice.

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References:

[1] R. Belohlavek, "Fuzzy Galois connections," Math. Logic Quarterly, 45, (4), pp. 497-504, 1999.
[2] R. Belohlavek, "Reduction and a simple proof of characterization of fuzzy concept lattices," Fundamenta Informaticae, 46, (4), pp. 277-285, 2001.
[3] R. Belohlavek, "Fuzzy Relational Systems: Foundations and Principles," Kluwer, Academic/Plenum Publishers, New York, 2002.
[4] R. Belohlavek, "Concept lattices and order in fuzzy logic," Ann. Pure Appl. Logic, 128, pp. 277-298, 2004.
[5] R. Belohlavek and T. Funiokova, "Fuzzy interior operators," Int. J. General Systems, 33, (4), pp. 315-330, 2004.
[6] R. Belohlavek, T. Funioková, and V. Vychodil, "Galois connections with hedges," In: Proc. 11th IFSA Congress 2005, pp. 1250-1255, Springer.
[7] R. Belohlavek, V. Sklenar, and J. Zacpal, "Crisply Generated Fuzzy Concepts," In: B. Ganter and R. Godin (Eds.), ICFCA 2005, LNCS, 3403, pp. 268-283, Springer-Verlag, Berlin/Heidelberg, 2005.
[8] R. Belohlavek and V. Vychodil, "Reducing the size of fuzzy concept lattices by hedges," In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22-25, 2005, Reno (Nevada, USA), pp. 663-668.
[9] R. Belohlavek and V. Vychodil, "What is a fuzzy concept lattice?," In: Proc. CLA 2005, 3rd Int. Conference on Concept Lattices and Their Applications, September 7-9, 2005, Olomouc, Czech Republic, pp. 34-45, URL: http://ceur-ws.org/Vol-162/.
[10] R. Belohlavek and V. Vychodil, "Attribute implications in a fuzzy setting," In: R. Missaoui and J. Schmid (Eds.), ICFCA 2006, LNAI, 3874, pp. 45-60, 2006.
[11] S. B. Yahia and A. Jaoua, "Discovering knowledge from fuzzy concept lattice," In: A. Kandel, M. Last, and H. Bunke, "Data Mining and Computational Intelligence," pp. 167-190, Physica-Verlag, 2001.
[12] A. Burusco and R. Fuentes-Gonzáles, "The study of the L-fuzzy concept lattice," Mathware & Soft Computing, 3, pp. 209-218, 1994.
[13] C. Carpineto and G. Romano, "Concept Data Analysis," Theory and Applications, J. Wiley, 2004.
[14] B. Ganter and R. Wille, "Formal Concept Analysis," Mathematical Foundations, Springer, Berlin, 1999.
[15] G. Gerla, "Fuzzy Logic," Mathematical Tools for Approximate Reasoning, Kluwer, Dordrecht, 2001.
[16] J. A. Goguen, "The logic of inexact concepts," Synthese, 18, pp. 325-373, 1968-9.
[17] P. Hájek, "Metamathematics of Fuzzy Logic," Kluwer, Dordrecht, 1998.
[18] P. Hájek, "On very true," Fuzzy Sets and Systems, 124, pp. 329-333, 2001.
[19] G. J. Klir and B. Yuan, "Fuzzy Sets and Fuzzy Logic," Theory and Applications, Prentice Hall, 1995.
[20] S. Krajči, "Cluster based efficient generation of fuzzy concepts," Neural Network World, 5, pp. 521-530, 2003.
[21] S. Krajči, "A generalized concept lattice," Logic Journal of IGPL, 13, (5), pp. 543-550, 2005.
[22] D. Maier, "The Theory of Relational Databases," Computer Science Press, Rockville, 1983.
[23] O. Ore, "Galois connections," Trans. Amer. Math. Soc., 55, pp. 493-513, 1944.
[24] S. Pollandt, "Fuzzy Begriffe," Springer, Berlin, 1997.
[25] G. Snelting and F. Tip, "Understanding class hierarchies using concept analysis," ACM Trans. Program. Lang. Syst., 22, (3), pp. 540-582, May 2000.

[26] G. Takeuti and S. Titani, "Globalization of intuitionistic set theory," Annals of Pure and Applied Logic, 33, pp. 195-211, 1987.

[27] L. A. Zadeh, "The concept of a linguistic variableand its application to approximate reasoning I, II, III," Inf. Sci., 8, (3), pp. 199-251, 1975; pp. 301-357; 9, pp. 43-80, 1975.

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