Cut and Weakening in Fuzzy Horn Logic

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Abstract

We study generalized deduction rules of cut and weakening in the context of equational fragment of Pavelka-style fuzzy logic using complete residuated lattices as the structures of truth degrees. The deduction rules in question are parameterized by a truth stresser, an additional unary operation on the structure of truth degrees. It is shown that the deductive system of fuzzy Horn logic can be replaced by several equivalent systems which use cut and weakening instead of the monotony rule.

Keywords: cut, fuzzy equality, fuzzy logic, Horn logic, weakening

1 Introduction

The importance of equational reasoning in universal algebra and computer science has been widely recognized. At present, there are numerous results on properties of implicationally defined classes of algebras (a survey can be found in [18]) and proofs from equational and implicational theories, e.g. [13, 15, 18]. So far, the development of the equational proof theory has been focused mainly on the aspects related to reasoning with identities and implications between identities in the context of classical (two-valued) logic. In the classical sense, an identity $t \approx t'$ denotes that t equals t'. This crisp relationship might be, however, inappropriate when describing inherently vague systems. In such cases, it might be convenient to deal with relationships in terms of "being similar (to a certain degree)" instead of "being (fully) equal". It is then natural to investigate equational reasoning from the viewpoint of fuzzy logic in narrow sense [9, 10, 12].

The initial paper [2] on *fuzzy equational logic* (FEL) has introduced a syntactico-semantically complete calculus for reasoning with fuzzy sets of identities. In [5], we present so-called *fuzzy Horn logic* (FHL), which deals with generalized implications between (truth-weighted) identities, extending thus the results of [2]. Both FEL and FHL are developed in Pavelka-style [10, 12, 14], i.e. a fixed complete residuated lattice is used as the structure of truth degrees, and the completeness theorems say that the appropriately defined provability degree is equal to the degree of semantic consequence.

In the results on the classical Horn logic [13, 15, 18], the authors have used various deductive systems. The original deductive system of FHL, as introduced in [5], generalizes the system of [18]. The present paper aims at the problem of replacing the *monotony rule* of FHL by a couple of deduction rules which generalize the well-known rules of *cut* and *weakening.* This problem is especially interesting because (i) deduction rules of FHL involve truth-weighted formulas, (ii) the generalized Horn clauses of FHL have truth-weighted premises, and (iii) deduction rules of FHL are parameterized by so-called truth stresser [4, 11]. Thus, we might be interested, for instance, in the influence of a truth stresser on the semantic

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part of a generalized cut. This issue is completely hidden in the classical Horn logic. It is shown below that the original deductive system of FHL can be replaced by several equivalent systems which use generalized cut and weakening instead of the monotony rule.

In Section 2 we briefly summarize preliminaries and basic syntactic notions of FHL. Section 3 deals with the generalized rules of the cut and weakening and presents several deductive systems based on these rules. Finally, we introduce the semantics of FHL and show a connection of the alternative deductive systems to FHL by presenting the completeness theorem.

2 Preliminaries

We use complete residuated lattices as the structures of truth degrees. A (*complete*) *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ of type $\langle 2,2,2,2,0,0\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (iii) $\langle \otimes, \rightarrow\rangle$ is an *adjoint pair,* i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$ (so called *adjointness property*). For $a \in L$, let a^n ($n \in \mathbb{N}_0$) denote the *n-th power of a,* i.e. $a^0 = 1$, $a^n = a \otimes a^{n-1}$ ($n > 0$). A mapping ${}^*\colon L \rightarrow L$ satisfying

$$1^* = 1, \tag{2.1}$$

$$a^* \leq a, \tag{2.2}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{2.3}$$

is called a *truth stresser* [11, 4, 5]. A complete residuated lattice $\mathbf{L}$ equipped with a truth stresser will be denoted by $\mathbf{L}^*$. Note that (2.1)–(2.3) ensure the *monotony*, i.e. $a \leq b$ implies $a^* \leq b^*$ ($a, b \in L$). Two extreme cases of truth stressers are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) so-called globalization [16]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

If $\mathbf{L}$ is a chain, the globalization coincides with Baaz's operation [1, 10]. In FHL, truth stresser influences both the syntactic and semantic consequence. Namely, on the syntax level, it is used as a thresholding function for deduction rules while on the semantics level, it determines the interpretation of implications between identities.

An $\mathbf{L}$-*set* A (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set U is a mapping $A\colon U \rightarrow L$, $A(u) \in L$ being interpreted as the truth degree of "element u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in universe U. For every $\mathbf{L}$-set $A\colon U \rightarrow L$, the *support set of A,* denoted by $\mathrm{Supp}(A)$, is defined by $\mathrm{Supp}(A) = \{u \in U \mid A(u) > 0\}$. An $\mathbf{L}$-set A is called *finite* if $\mathrm{Supp}(A)$ is finite. An $\mathbf{L}$-set A with $\mathrm{Supp}(A) \subseteq \{u\}$, denoted by $\{{}^{A(u)}/u\}$, is called a *singleton.* Basic operations with $\mathbf{L}$-sets are defined componentwise using operations of $\mathbf{L}$ [3]. A *binary* $\mathbf{L}$-*relation* R on U is an $\mathbf{L}$-set in the universe set $U \times U$, i.e. a mapping $R : U \times U \rightarrow L$. For $\mathbf{L}$-sets A and B on the same universe set U we write $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$; and $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Now we introduce basic syntactic notions of fuzzy Horn logic [5]. A *type* is a collection F of function symbols, each with its arity. Given a complete residuated lattice $\mathbf{L}$, the language of FHL consists of (at least denumerable) set X of variables, a type F, a binary predicate symbol $\approx$ standing for (fuzzy) equality, a set $\{\overline{a} \mid a \in L\}$ of symbols of truth degrees (however, for brevity and since there is no danger of confusion, we identify $\overline{a}$ with a), and symbols of logical connectives $\Rightarrow$ (implication) and $\overline{\wedge}$ (conjunction). Terms and identities are denoted by $t, s, \ldots$ and $t \approx t', s \approx s', \ldots$, respectively, both possibly with indices. Let $T(X)$ denote the set of all terms of type F over X. A *Horn clause* (*with truth-weighted premises*), denoted by $P \Rightarrow (t \approx t')$, is a syntactic expression of the form

$$\langle t_1 \approx t'_1, P(t_1, t'_1)\rangle \mathbin{\overline{\wedge}} \cdots \mathbin{\overline{\wedge}} \langle t_n \approx t'_n, P(t_n, t'_n)\rangle \Rightarrow (t \approx t'),$$

where P is a finite binary $\mathbf{L}$-relation on $T(X)$ with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$, and $t, t' \in T(X)$. Horn clauses are the basic formulas of FHL. Note that P can be thought of as an $\mathbf{L}$-set of premises, and $P(s, s') \in L$ can be interpreted as a degree (weight) to which an identity $s \approx s'$ belongs to the $\mathbf{L}$-set of premises P. Horn clause $P \Rightarrow (t \approx t')$ with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$ can be read as follows: "If t_1 equals t'_1 in degree (at least) $P(t_1, t'_1)$, and $\cdots$ and t_n equals t'_n in degree (at least) $P(t_n, t'_n)$, then t equals t'". For more details we refer to [5, 6].

Given a Horn clause φ and a truth degree $a \in L$, the couple $\langle \varphi, a\rangle$ is called a *weighted Horn clause.* Weighted Horn clauses will be denoted by $\langle P \Rightarrow (t \approx t'), a\rangle$. For brevity, we write $\langle t \approx t', a\rangle$ instead of $\langle \emptyset \Rightarrow (t \approx t'), a\rangle$. $\mathbf{L}$-sets of Horn clauses are denoted usually by $\Gamma, \Sigma, \ldots$ Let *Fml* denote the set of all Horn clauses. Every partial mapping $R\colon (\mathit{Fml} \times L)^n \to \mathit{Fml} \times L$ is called an $\mathbf{L}^*$-deduction rule. Nullary $\mathbf{L}^*$-deduction rules will be called *axioms.* For convenience, instead of $R(\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle) = \langle \varphi, a\rangle$, we write

$$R\colon\ \frac{\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle}{\langle \varphi, a\rangle}.$$

On the verbal level, $R(\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle) = \langle \varphi, a\rangle$ should be read as: "from φ_1 in degree a_1, and $\cdots$ and φ_n in degree a_n infer φ in degree a". A system $\mathcal{R}$ of $\mathbf{L}^*$-deduction rules is called an $\mathbf{L}^*$-deductive system. Let Γ be an $\mathbf{L}$-set of Horn clauses and let $\mathcal{R}$ be an $\mathbf{L}^*$-deductive system. An ($\mathbf{L}^*$*-weighted*) $\mathcal{R}$*-proof* of $\langle P \Rightarrow (t \approx t'), a\rangle$ from Γ, is a finite sequence of weighted Horn clauses $\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_l, a_l\rangle$, where φ_l is $P \Rightarrow (t \approx t')$, $a_l = a$, and for every $\langle \varphi_i, a_i\rangle$ $(i = 1, \ldots, l)$, we have either $a_i = \Gamma(\varphi_i)$ or there is an n-ary $\mathbf{L}^*$-deduction rule $R \in \mathcal{R}$ such that $R(\langle \varphi_{i_1}, a_{i_1}\rangle, \ldots, \langle \varphi_{i_n}, a_{i_n}\rangle) = \langle \varphi_i, a_i\rangle$ for some $i_1, \ldots, i_n < i$. A weighted Horn clause $\langle P \Rightarrow (t \approx t'), b\rangle$ is said to be $\mathcal{R}$*-provable from* Γ*,* denoted by $\Gamma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), b\rangle$, if there is an $\mathcal{R}$-proof of $\langle P \Rightarrow (t \approx t'), b\rangle$ from Γ. A Horn clause $P \Rightarrow (t \approx t')$ is called $\mathcal{R}$-*provable from* Γ *in degree* (*at least*) $b \in L$, if $\Gamma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), b\rangle$. For every Horn clause $P \Rightarrow (t \approx t')$ we define a *degree* $|P \Rightarrow (t \approx t')|^{\mathcal{R}}_{\Gamma}$ *of* $\mathcal{R}$*-provability of* $P \Rightarrow (t \approx t')$ *from* Γ by $|P \Rightarrow (t \approx t')|^{\mathcal{R}}_{\Gamma} = \bigvee\{a \in L \mid \Gamma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), a\rangle\}$.

The following schemes of $\mathbf{L}^*$-deduction rules were introduced in [5]:

(Ref): $\langle P \Rightarrow (t \approx t), 1\rangle$,

(Sym): $\dfrac{\langle P \Rightarrow (t \approx t'), a\rangle}{\langle P \Rightarrow (t' \approx t), a\rangle}$,

$$\text{(Tra)}: \frac{\langle P \Rightarrow (t \approx t'), a\rangle,\ \langle P \Rightarrow (t' \approx t''), b\rangle}{\langle P \Rightarrow (t \approx t''), a \otimes b\rangle},$$

$$\text{(Rep)}: \frac{\langle P \Rightarrow (t \approx t'), a\rangle}{\langle P \Rightarrow (s \approx s'), a\rangle},$$

$$\text{(Ext)}: \langle P \Rightarrow (t \approx t'), P(t,t')\rangle,$$

$$\text{(Sub)}: \frac{\langle P \Rightarrow (t \approx t'), a\rangle}{\big\langle P(x/r) \Rightarrow \big(t(x/r) \approx t'(x/r)\big), a\big\rangle},$$

$$\text{(Mon)}: \frac{\{\langle Q \Rightarrow (t_i \approx t'_i), a_i\rangle;\, i = 1,\dots,n\},\ \langle P \Rightarrow (t \approx t'), b\rangle}{\big\langle Q \Rightarrow (t \approx t'), b \otimes \bigotimes_{i=1}^{n}(P(t_i,t'_i) \rightarrow a_i)^*\big\rangle},$$

$$\text{(Sup)}: \frac{\langle P \Rightarrow (t \approx t'), a\rangle,\ \langle P \Rightarrow (t \approx t'), b\rangle}{\langle P \Rightarrow (t \approx t'), a \vee b\rangle},$$

where $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \dots, \langle t_n, t'_n\rangle\}$, $s, s' \in T(X)$, $x \in X$; $a, b, a_1, \dots, a_n \in L$, s contains t as a subterm and s' results from s by replacing one occurrence of t in s by t'; $t(x/r)$ denotes the term resulting from t by substitution of all occurrences of x by $r \in T(X)$; $P(x/r)$ is a finite binary **L**-relation on $T(X)$ defined by $\big(P(x/r)\big)(t,t') = \bigvee \{P(s,s') \mid s(x/r) = t \text{ and } s'(x/r) = t'\}$ for all $t, t' \in T(X)$.

Remark 2.1
(1) We use the following convention for specifying $\mathbf{L}^*$-deductive systems: we put $\mathcal{R} = \{\dots, \text{(Ref)}, \dots\}$ to denote that $\mathcal{R}$ consists of $\dots, \text{(Ref)}, \dots$ Until otherwise mentioned, $\mathcal{R}$ always stands for an $\mathbf{L}^*$-deductive system defined by $\mathcal{R} = \{\text{(Ref)}, \text{(Sym)}, \text{(Tra)}, \text{(Rep)}, \text{(Ext)}, \text{(Sub)}, \text{(Mon)}, \text{(Sup)}\}$.

(2) Note that unlike the deduction rules of Pavelka-style logic [10, 12, 14], we do not separate syntactic and semantic part of a rule by two independent (partial) mappings. This is, however, only for the sake of convenience. Namely, in [5] we showed that all of our deduction rules are in fact derived rules in a suitably extended Pavelka-style first-order fuzzy logic. This applies also to all the rules introduced in this paper. Moreover, each of the above-introduced rules (i.e., (Ref), (Sym), ..., (Sup)) is *nondecreasing* in the sense that from $a_1 \leq a'_1, \dots, a_n \leq a'_n$, $R(\langle \varphi_1, a_1\rangle, \dots, \langle \varphi_n, a_n\rangle) = \langle \varphi, a\rangle$, and $R(\langle \varphi_1, a'_1\rangle, \dots, \langle \varphi_n, a'_n\rangle) = \langle \varphi, a'\rangle$, it follows that $a \leq a'$.

Definition 2.2
For $\mathbf{L}^*$-deductive systems $\mathcal{R}_1, \mathcal{R}_2$ we put $\mathcal{R}_1 \leq \mathcal{R}_2$ if $|\varphi|_\Gamma^{\mathcal{R}_1} \leq |\varphi|_\Gamma^{\mathcal{R}_2}$ for every Horn clause φ, and every **L**-set Γ of Horn clauses. $\mathbf{L}^*$-deductive systems $\mathcal{R}_1, \mathcal{R}_2$ are said to be *equivalent*, denoted by $\mathcal{R}_1 \equiv \mathcal{R}_2$, if $\mathcal{R}_1 \leq \mathcal{R}_2$ and $\mathcal{R}_2 \leq \mathcal{R}_1$.

Obviously, $\mathcal{R}_1 \equiv \mathcal{R}_2$ iff $|\varphi|_\Gamma^{\mathcal{R}_1} = |\varphi|_\Gamma^{\mathcal{R}_2}$ for every Horn clause φ, and every **L**-set Γ of Horn clauses.

3 Deductive systems based on cut and weakening

First, we pay our attention to the $\mathbf{L}^*$-deduction rule of *monotony* (Mon) and show its relationship to generalized rules of *cut* and *weakening.* Later on, we present an $\mathbf{L}^*$-deductive system which is equivalent to $\mathcal{R}$, but unlike $\mathcal{R}$, it contains more axioms and simpler $\mathbf{L}^*$-deduction rules.

In the sequel, we consider the following $\mathbf{L}^*$-deduction rules:

$$\text{(Cut)}\colon \frac{\langle Q \Rightarrow (s \approx s'), a\rangle, \langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle}{\big\langle Q \cup P \Rightarrow (t \approx t'), b \otimes ((c \vee P(s,s')) \to a)^{*}\big\rangle},$$

$$\text{(Wea)}\colon \frac{\langle P \Rightarrow (t \approx t'), a\rangle}{\langle Q \cup P \Rightarrow (t \approx t'), a\rangle}.$$

Remark 3.1
(1) In the bivalent case, i.e. for $\mathbf{L}$ being the two-element Boolean algebra, (Cut) and (Wea) coincide with their ordinary counterparts. Clearly, for $a = b = c = 1$, and denoting $\{{}^{1}/\langle s, s'\rangle\}$ by $\{s \approx s'\}$, we obtain the following classical deduction rules

$$\frac{Q \Rightarrow (s \approx s'), \{s \approx s'\} \cup P \Rightarrow (t \approx t')}{Q \cup P \Rightarrow (t \approx t')}, \qquad \frac{P \Rightarrow (t \approx t')}{Q \cup P \Rightarrow (t \approx t')}.$$

(2) Note that $\{{}^{c}/\langle s, s'\rangle\} \cup P$ used in (Cut) represents an $\mathbf{L}$-set of premises P extended by an identity $s \approx s'$ in the truth degree $c \in L$. The aspect of extending sets of premises by identities in truth degrees other than 0 and 1 is entirely hidden if we consider classical formulas. On the verbal level, (Cut) can be read: "if Q implies $s \approx s'$ and if P extended by $s \approx s'$ in degree c implies $t \approx t'$, then $Q \cup P$ implies $t \approx t'$". A finer reading of (Cut) is "$Q \cup P$ implies $t \approx t'$ (at least) in degree to which P extended by $s \approx s'$ in degree c implies $t \approx t'$ *and* $s \approx s'$ is implied by Q at least in degree to which $s \approx s'$ belongs to P extended by $s \approx s'$ in degree c".

(3) The (Cut) rule is parameterized by * similarly as (Mon). That is, given $\mathbf{L}$, different *'s lead to different interpretations of (Cut). For instance, if * is the globalization, i.e. it is defined by (2.4), (Cut) is equivalent to

$$\frac{\langle Q \Rightarrow (s \approx s'), a\rangle, \langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle}{\langle Q \cup P \Rightarrow (t \approx t'), b\rangle} \quad \text{if } c \leq a \text{ and } P(s,s') \leq a.$$

Indeed, (Cut) is used in a nontrivial way iff $((c \vee P(s,s')) \to a)^* = 1$, i.e. iff $c \vee P(s,s') \leq a$, which is equivalent to $c \leq a$ and $P(s,s') \leq a$. In such a case, the resulting formula is inferred in degree b. In the other case, the truth degree of the resulting formula equals 0 (not interesting from the provability standpoint). Thus, the globalization can be seen as a *threshold function* which disallows (Cut) to infer the resulting formula in a nonzero truth degree if the truth degree $a \in L$ of the input formula does not exceed the threshold values $c \in L$ and $P(s,s') \in L$.

Lemma 3.2
Let $\mathcal{R}_{\mathrm{E,M}} = \{(\mathrm{Ext}), (\mathrm{Mon})\}$, $\mathcal{R}_{\mathrm{W}} = \{(\mathrm{Wea})\}$, and $\mathcal{R}_{\mathrm{E,M,W}} = \{(\mathrm{Ext}), (\mathrm{Mon}), (\mathrm{Wea})\}$. Then $\mathcal{R}_{\mathrm{W}} \leq \mathcal{R}_{\mathrm{E,M}}$ and $\mathcal{R}_{\mathrm{E,M,W}} \equiv \mathcal{R}_{\mathrm{E,M}}$.

Proof. "$\mathcal{R}_{\mathrm{W}} \leq \mathcal{R}_{\mathrm{E,M}}$": Let Γ be an $\mathbf{L}$-set of Horn clauses, $\langle Q \Rightarrow (t \approx t'), a\rangle$ be a member of an $\mathcal{R}_{\mathrm{W}}$-proof inferred from $\langle P \Rightarrow (t \approx t'), a\rangle$ using (Wea). Thus, $P \subseteq Q$. Let $\mathrm{Supp}(P) = \{\langle t_1, t_1'\rangle, \ldots, \langle t_n, t_n'\rangle\}$. Using induction hypothesis, there is an $\mathcal{R}_{\mathrm{E,M}}$-proof

$\delta_1, \dots, \delta_l, \langle P \Rightarrow (t \approx t'), b\rangle$ with $a \leq b$. The sequence

$$
\begin{array}{lll}
1: & \langle Q \Rightarrow (t_1 \approx t'_1), Q(t_1, t'_1)\rangle, & \text{by (Ext)} \\
 & \vdots & \vdots \\
n: & \langle Q \Rightarrow (t_n \approx t'_n), Q(t_n, t'_n)\rangle, & \text{by (Ext)} \\
 & \delta_1, \dots, \delta_l, & \\
n+1: & \langle P \Rightarrow (t \approx t'), b\rangle, & \text{proof of } \langle P \Rightarrow (t \approx t'), b\rangle \\
n+2: & \langle Q \Rightarrow (t \approx t'), c\rangle, & \text{by (Mon) on } 1, \dots, n, n+1
\end{array}
$$

where $c = b \otimes \bigotimes_{i=1}^{n} (P(t_i, t'_i) \to Q(t_i, t'_i))^* = b \otimes \bigotimes_{i=1}^{n} 1^* = b$, is an $\mathcal{R}_{\mathrm{E,M}}$-proof of $\langle Q \Rightarrow (t \approx t'), c\rangle$ from Γ, showing $\Gamma \vdash^{\mathcal{R}_{\mathrm{E,M}}} \langle Q \Rightarrow (t \approx t'), b\rangle$ with $b \geq a$. The rest is obvious. ∎

Lemma 3.3
Let $\mathcal{R}_{\mathrm{E,M}} = \{(\mathrm{Ext}), (\mathrm{Mon})\}$ and $\mathcal{R}_{\mathrm{C}} = \{(\mathrm{Cut})\}$. Then $\mathcal{R}_{\mathrm{C}} \leq \mathcal{R}_{\mathrm{E,M}}$.

Proof. Let $\langle Q \cup P \Rightarrow (t \approx t'), d\rangle$ be a member of an $\mathcal{R}_{\mathrm{C}}$-proof derived by (Cut) from weighted Horn clauses $\langle Q \Rightarrow (s \approx s'), a\rangle$ and $\langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle$. Thus, $d = b \otimes ((c \vee P(s, s')) \to a)^*$. Suppose, we have $\mathrm{Supp}(P) - \{\langle s, s'\rangle\} = \{\langle t_1, t'_1\rangle, \dots, \langle t_n, t'_n\rangle\}$. By induction hypothesis, there are $\mathcal{R}_{\mathrm{E,M}}$-proofs of weighted Horn clauses $\langle Q \Rightarrow (s \approx s'), a'\rangle$ and $\langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b'\rangle$ such that $a \leq a'$ and $b \leq b'$. Assuming $c \vee P(s, s') \neq 0$, we can consider the sequence

$$
\begin{array}{lll}
1: & \langle Q \cup P \Rightarrow (t_1 \approx t'_1), (Q \cup P)(t_1, t'_1)\rangle, & \text{by (Ext)} \\
 & \vdots & \vdots \\
n: & \langle Q \cup P \Rightarrow (t_n \approx t'_n), (Q \cup P)(t_n, t'_n)\rangle, & \text{by (Ext)} \\
 & \delta_1, \dots, \delta_l, & \\
n+1: & \langle Q \Rightarrow (s \approx s'), a'\rangle, & \text{proof of } \langle Q \Rightarrow (s \approx s'), a'\rangle \\
n+2: & \langle Q \cup P \Rightarrow (s \approx s'), a'\rangle, & \text{by (Wea) on } n+1 \\
 & \delta'_1, \dots, \delta'_{l'}, & \\
n+3: & \langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b'\rangle, & \text{proof of } \langle \{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b'\rangle \\
n+4: & \langle Q \cup P \Rightarrow (t \approx t'), d'\rangle, & \text{by (Mon) on } 1, \dots, n, n+2, n+3
\end{array}
$$

where

$$
\begin{aligned}
d' &= b' \otimes \bigotimes_{i=1}^{n} ((\{{}^{c}/\langle s, s'\rangle\} \cup P)(t_i, t'_i) \to (Q \cup P)(t_i, t'_i))^* \otimes ((c \vee P(s, s')) \to a')^* = \\
&= b' \otimes \bigotimes_{i=1}^{n} (P(t_i, t'_i) \to (Q \cup P)(t_i, t'_i))^* \otimes ((c \vee P(s, s')) \to a')^* = \\
&= b' \otimes ((c \vee P(s, s')) \to a')^*.
\end{aligned}
$$

The above sequence is an $\mathcal{R}_{\mathrm{E,M,W}}$-proof. Hence, assuming $c \vee P(s, s') \neq 0$, we get that $Q \cup P \Rightarrow (t \approx t')$ is $\mathcal{R}_{\mathrm{E,M}}$-provable in degree at least d' due to Lemma 3.2. In addition to that, $d = b \otimes ((c \vee P(s, s')) \to a)^* \leq b' \otimes ((c \vee P(s, s')) \to a')^* = d'$ since $\otimes$ and * are monotone, and $\to$ is isotone in the second argument. If $c \vee P(s, s') = 0$, we can proceed analogously as above provided that we skip steps "n+1" and "n+2" in the previous proof. ∎

Lemma 3.4
Let $\mathcal{R}_{\mathrm{M}} = \{(\mathrm{Mon})\}$, $\mathcal{R}_{\mathrm{C,E,W}} = \{(\mathrm{Cut}), (\mathrm{Ext}), (\mathrm{Wea})\}$, and $\mathcal{R}_{\mathrm{C,E,S,W}} = \{(\mathrm{Cut}), (\mathrm{Ext}), (\mathrm{Sup}), (\mathrm{Wea})\}$. Then $\mathcal{R}_{\mathrm{M}} \leq \mathcal{R}_{\mathrm{C,E,S,W}}$. Moreover, if $\mathbf{L}$ is a chain then $\mathcal{R}_{\mathrm{M}} \leq \mathcal{R}_{\mathrm{C,E,W}}$.

PROOF. Suppose $\langle Q \Rightarrow (t \approx t'), b\rangle$ is a member of an $\mathcal{R}_{\mathrm{M}}$-proof derived by (Mon) from $\langle P \Rightarrow (t \approx t'), b\rangle$ and $\langle Q \Rightarrow (t_i \approx t'_i), a_i\rangle$ $(i = 1, \ldots, n)$, where $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$. By induction hypothesis, we have $\Gamma \vdash^{\mathcal{R}_{\mathrm{C,E,S,W}}} \langle P \Rightarrow (t \approx t'), b'\rangle$ with $b \leq b'$, and $\Gamma \vdash^{\mathcal{R}_{\mathrm{C,E,S,W}}} \langle Q \Rightarrow (t_i \approx t'_i), a'_i\rangle$, where $a_i \leq a'_i$ $(i = 1, \ldots, n)$.

Fix $i \in \{1, \ldots, n\}$. The sequence

	$\delta'_1, \ldots, \delta'_l,$	
1:	$\langle Q \Rightarrow (t_i \approx t'_i), a'_i\rangle,$	proof of $\langle Q \Rightarrow (t_i \approx t'_i), a'_i\rangle$
2:	$\langle Q \Rightarrow (t_i \approx t'_i), Q(t_i, t'_i)\rangle,$	by (Ext)
3:	$\langle Q \Rightarrow (t_i \approx t'_i), a'_i \vee Q(t_i, t'_i)\rangle,$	by (Sup) on $1, 2$

is an $\mathcal{R}_{\mathrm{C,E,S,W}}$-proof of $\langle Q \Rightarrow (t_i \approx t'_i), a'_i \vee Q(t_i, t'_i)\rangle$ from Γ. Hence, there is no loss of generality in assuming $Q(t_i, t'_i) \leq a'_i$ $(i = 1, \ldots, n)$. Let us introduce finite binary **L**-relations $P_0, \ldots, P_n$ on $T(X)$:

$$P_0 = P,$$
$$P_{i+1}(r, r') = \begin{cases} 0 & \text{for } r = t_{i+1},\ r' = t'_{i+1}, \\ P_i(r, r') & \text{otherwise.} \end{cases}$$

Evidently, $P_n(r, r') = 0$ $(r, r' \in T(X))$. Moreover, $P_i = \{{}^{P(t_{i+1}, t'_{i+1})}/\langle t_{i+1}, t'_{i+1}\rangle\} \cup P_{i+1}$ $(i = 1, \ldots, n)$, i.e. each $Q \cup P_i$ is of the form $\{{}^{P(t_{i+1}, t'_{i+1})}/\langle t_{i+1}, t'_{i+1}\rangle\} \cup (Q \cup P_{i+1})$ $(i = 1, \ldots, n)$. Thus, the following sequence

	$\delta'_1, \ldots, \delta'_{l'},$	
1:	$\langle P_0 \Rightarrow (t \approx t'), b'\rangle,$	proof of $\langle P \Rightarrow (t \approx t'), b'\rangle$
2:	$\langle Q \cup P_0 \Rightarrow (t \approx t'), b'\rangle,$	by (Wea) on 1
	$\delta_{1,1}, \ldots, \delta_{1,l_1},$	
3:	$\langle Q \Rightarrow (t_1 \approx t'_1), a'_1\rangle,$	proof of $\langle Q \Rightarrow (t_1 \approx t'_1), a'_1\rangle$, $Q(t_1, t'_1) \leq a'_1$
4:	$\langle Q \cup P_1 \Rightarrow (t_1 \approx t'_1), a'_1\rangle,$	by (Wea) on 3
5:	$\langle Q \cup P_1 \Rightarrow (t \approx t'), b_1\rangle,$	by (Cut) on $2, 4$
	$\delta_{2,1}, \ldots, \delta_{2,l_1},$	
6:	$\langle Q \Rightarrow (t_2 \approx t'_2), a'_2\rangle,$	proof of $\langle Q \Rightarrow (t_2 \approx t'_2), a'_2\rangle$, $Q(t_2, t'_2) \leq a'_2$
7:	$\langle Q \cup P_2 \Rightarrow (t_2 \approx t'_2), a'_2\rangle,$	by (Wea) on 6
8:	$\langle Q \cup P_2 \Rightarrow (t \approx t'), b_2\rangle,$	by (Cut) on $5, 7$
⋮	⋮	⋮
$3n{+}2$:	$\langle Q \cup P_n \Rightarrow (t \approx t'), b_n\rangle,$	by (Cut) on $3n{-}1, 3n{+}1$

is an $\mathcal{R}_{\mathrm{C,E,S,W}}$-proof of $\langle Q \Rightarrow (t \approx t'), b_n\rangle$ from Γ. We finish the proof by checking $b_j = b' \otimes \bigotimes_{i=1}^{j} (P(t_i, t'_i) \to a'_i)^*$ $(j = 1, \ldots, n)$. For $j = 1$, we have

$$\begin{aligned} b_1 &= b' \otimes \big(\big(P(t_1, t'_1) \vee (Q \cup P_1)(t_1, t'_1)\big) \to a'_1\big)^* = \\ &= b' \otimes \big(\big(P(t_1, t'_1) \vee Q(t_1, t'_1) \vee P_1(t_1, t'_1)\big) \to a'_1\big)^* = \\ &= b' \otimes \big((P(t_1, t'_1) \vee Q(t_1, t'_1)) \to a'_1\big)^* = \\ &= b' \otimes \big((P(t_1, t'_1) \to a'_1) \wedge (Q(t_1, t'_1) \to a'_1)\big)^* = \\ &= b' \otimes \big((P(t_1, t'_1) \to a'_1) \wedge 1\big)^* = b' \otimes (P(t_1, t'_1) \to a'_1)^*. \end{aligned}$$

Suppose the claim holds for j. It follows that

$$\begin{aligned} b_{j+1} &= b_j \otimes \big((P(t_{j+1},t'_{j+1}) \vee (Q \cup P_{j+1})(t_{j+1},t'_{j+1})) \to a'_{j+1}\big)^* = \\ &= \big(b' \otimes \textstyle\bigotimes_{i=1}^{j} (P(t_i,t'_i) \to a'_i)^*\big) \otimes \big(P(t_{j+1},t'_{j+1}) \to a'_{j+1}\big)^* = \\ &= b' \otimes \textstyle\bigotimes_{i=1}^{j+1} (P(t_i,t'_i) \to a'_i)^*, \end{aligned}$$

yielding $\Gamma \vdash^{\mathcal{R}_{\mathrm{C,E,S,W}}} \big\langle Q \Rightarrow (t \approx t'), b' \otimes \bigotimes_{i=1}^{n} (P(t_i,t'_i) \to a'_i)^*\big\rangle$. Monotony of $\otimes$ and * together with isotony of $\to$ in the second argument ensure $b' \otimes \bigotimes_{i=1}^{n} (P(t_i,t'_i) \to a'_i)^* \geq b \otimes \bigotimes_{i=1}^{n} (P(t_i,t'_i) \to a_i)^*$. If $\mathbf{L}$ is a chain then obviously $\mathcal{R}_{\mathrm{C,E,S,W}} \equiv \mathcal{R}_{\mathrm{C,E,W}}$, finishing the proof. ∎

THEOREM 3.5
Let $\mathcal{R}_{\mathrm{G}}$ result from $\mathcal{R}$ by replacing (Mon) by (Cut) and (Wea). Then $\mathcal{R}_{\mathrm{G}} \equiv \mathcal{R}$.

PROOF. Consequence of Lemma 3.2, Lemma 3.3, and Lemma 3.4. ∎

Let us note that in the classical Horn logic [13, 15, 18], the rule of replacement can be substituted by so-called rule of *compatibility* (*congruence*). In fuzzy setting, such a rule is naturally formalized as follows

$$(\mathrm{Com})\colon \frac{\langle P \Rightarrow (t_1 \approx t'_1), a_1\rangle, \ldots, \langle P \Rightarrow (t_n \approx t'_n), a_n\rangle}{\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,\ldots,t'_n)), a_1 \otimes \cdots \otimes a_n\rangle},$$

where $t_1, t'_1, \ldots, t_n, t'_n \in T(X)$, $a_1, \ldots, a_n \in L$, and $f \in F$ is an n-ary function symbol. We have

LEMMA 3.6
Let $\mathcal{R}'$ result from $\mathcal{R}$ by replacing (Rep) by (Com). Then $\mathcal{R}' \equiv \mathcal{R}$.

PROOF. Suppose an $\mathbf{L}$-set of Horn clauses Γ is given. We examine only the cases when (Com) and (Rep) are used since the rest is evident.

"$\mathcal{R}' \leq \mathcal{R}$": Let $\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,\ldots,t'_n)), a\rangle$ be a member of an $\mathcal{R}'$-proof inferred from weighted Horn clauses $\langle P \Rightarrow (t_1 \approx t'_1), a_1\rangle, \ldots, \langle P \Rightarrow (t_n \approx t'_n), a_n\rangle$ by (Com). Thus, $a = a_1 \otimes \cdots \otimes a_n$ and by induction hypothesis, $\Gamma \vdash^{\mathcal{R}} \langle P \Rightarrow (t_1 \approx t'_1), b_1\rangle, \ldots, \Gamma \vdash^{\mathcal{R}} \langle P \Rightarrow (t_n \approx t'_n), b_n\rangle$ with $a_i \leq b_i$ $(i = 1, \ldots, n)$. Hence,

	$\delta_{1,1}, \ldots, \delta_{1,l_1}$,	
1:	$\langle P \Rightarrow (t_1 \approx t'_1), b_1\rangle$,	proof of $\langle P \Rightarrow (t_1 \approx t'_1), b_1\rangle$
$\vdots$	$\vdots$	$\vdots$
	$\delta_{n,1}, \ldots, \delta_{n,l_n}$,	
n:	$\langle P \Rightarrow (t_n \approx t'_n), b_n\rangle$,	proof of $\langle P \Rightarrow (t_n \approx t'_n), b_n\rangle$
$n+1$:	$\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,t_2,\ldots,t_n)), b_1\rangle$,	by (Rep) on 1
$n+2$:	$\langle P \Rightarrow (f(t'_1,t_2,\ldots,t_n) \approx f(t'_1,t'_2,t_3,\ldots,t_n)), b_2\rangle$,	by (Rep) on 2
$\vdots$	$\vdots$	$\vdots$
$2n$:	$\big\langle P \Rightarrow (f(t'_1,\ldots,t'_{n-1},t_n) \approx f(t'_1,\ldots,t'_n)), b_n\big\rangle$,	by (Rep) on n
$2n+1$:	$\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,t'_2,t_3,\ldots,t_n)), b_1 \otimes b_2\rangle$,	by (Tra)
$\vdots$	$\vdots$	$\vdots$
$3n-1$:	$\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,\ldots,t'_n)), b_1 \otimes \cdots \otimes b_n\rangle$	by (Tra)

is an $\mathcal{R}$-proof of $\langle P \Rightarrow (f(t_1,\ldots,t_n) \approx f(t'_1,\ldots,t'_n)), b\rangle$ from Γ with $a \leq b_1 \otimes \cdots \otimes b_n = b$, showing "$\leq$".

"$\mathcal{R} \leq \mathcal{R}'$": Let $\langle P \Rightarrow (s \approx s'), a\rangle$ be a member of an $\mathcal{R}$-proof such that weighted Horn clause $\langle P \Rightarrow (s \approx s'), a\rangle$ results from $\langle P \Rightarrow (t \approx t'), a\rangle$ by (Rep). We proceed by structural induction. For s being t, the claim is trivial. Thus, let $s = f(t_1,\ldots,t_n)$ and $s' = f(t_1,\ldots,t_{k-1},t'_k,t_{k+1},\ldots,t_n)$, where t_k has t as a subterm and t'_k results from t_k by replacing of one occurrence of t by t'. By induction hypothesis, $\Gamma \vdash^{\mathcal{R}'} \langle P \Rightarrow (t_k \approx t'_k), b\rangle$, where $a \leq b$, i.e. there is an $\mathcal{R}'$-proof $\delta_1,\ldots,\delta_l,\langle P \Rightarrow (t_k \approx t'_k), b\rangle$ from Γ. The following sequence

1:	$\langle P \Rightarrow (t_1 \approx t_1), 1\rangle$,	by (Ref)
$\vdots$	$\vdots$	$\vdots$
$k-1$:	$\langle P \Rightarrow (t_{k-1} \approx t_{k-1}), 1\rangle$,	by (Ref)
	$\delta_1,\ldots,\delta_l$,	
k:	$\langle P \Rightarrow (t_k \approx t'_k), b\rangle$,	proof of $\langle P \Rightarrow (t_k \approx t'_k), b\rangle$
$k+1$:	$\langle P \Rightarrow (t_{k+1} \approx t_{k+1}), 1\rangle$,	by (Ref)
$\vdots$	$\vdots$	$\vdots$
n:	$\langle P \Rightarrow (t_n \approx t_n), 1\rangle$,	by (Ref)
$n+1$:	$\left\langle P \Rightarrow (s \approx s'), 1^{k-1} \otimes b \otimes 1^{n-k}\right\rangle$	by (Com) on $1,\ldots,n$

is an $\mathcal{R}'$-proof of $\langle P \Rightarrow (s \approx s'), b\rangle$ from Γ. Thus, $\mathcal{R} \leq \mathcal{R}'$. ∎

Now we shall show that some of the n-ary deduction rules of $\mathcal{R}$ can be substituted by axioms, i.e. nullary deduction rules. The reduction of n-ary deduction rules presented here is inspired by the Gentzen-style axiomatizations of (classical) equational logic introduced in [13]. The problem of representing deduction rules by Horn clauses is especially interesting in fuzzy setting because the $\mathbf{L}^*$-deduction rules involve truth-weighted formulas.

Let us have the following axioms:

(ARef): $\langle t \approx t, 1\rangle$,

(ASym): $\langle\langle t \approx t', a\rangle \Rightarrow (t' \approx t), a\rangle$,

(ATra): $\langle\langle t \approx t', a\rangle \barwedge \langle t' \approx t'', b\rangle \Rightarrow (t \approx t''), a \otimes b\rangle$,

(ARep): $\langle\langle t \approx t', a\rangle \Rightarrow (s \approx s'), a\rangle$,

where $t, t', t'' \in T(X)$ are arbitrary, s' is a term resulting from $s \in T(X)$ by substitution of one occurrence of t in s by t', and $a, b \in L$.

Lemma 3.7

Let $\mathcal{R}_{\mathrm{A}}$ result from $\mathcal{R}$ by replacing (Ref)–(Rep) by (ARef)–(ARep). Then $\mathcal{R}_{\mathrm{A}} \equiv \mathcal{R}$.

Proof. Take an $\mathbf{L}$-set Γ of Horn clauses.

"$\mathcal{R}_{\mathrm{A}} \leq \mathcal{R}$": It suffices to check that each axiom (ARef)–(ARep) is $\mathcal{R}$-provable.

(ARef): Evidently, applying (Ref) on identity $t \approx t$ (i.e. a Horn clause $\emptyset \Rightarrow (t \approx t)$) gives $\langle t \approx t, 1\rangle$.

(ASym): $\Gamma \vdash^{\mathcal{R}} \langle\langle t \approx t', a\rangle \Rightarrow (t \approx t'), a\rangle$ due to (Ext).
Thus, (Sym) gives $\Gamma \vdash^{\mathcal{R}} \langle\langle t \approx t', a\rangle \Rightarrow (t' \approx t), a\rangle$.

(ATra): $\Gamma \vdash^{\mathcal{R}} \langle\langle t \approx t', a\rangle \barwedge \langle t' \approx t'', b\rangle \Rightarrow (t \approx t'), a\rangle$, and
$\Gamma \vdash^{\mathcal{R}} \langle\langle t \approx t', a\rangle \barwedge \langle t' \approx t'', b\rangle \Rightarrow (t' \approx t''), b\rangle$ on account of (Ext). By applying (Tra), we can conclude $\Gamma \vdash^{\mathcal{R}} \langle\langle t \approx t', a\rangle \barwedge \langle t' \approx t'', b\rangle \Rightarrow (t \approx t''), a \otimes b\rangle$.

(ARep): Proceed similarly as for (ASym).

"$\mathcal{R} \leq \mathcal{R}_{\mathrm{A}}$": Using induction on the length of an $\mathcal{R}$-proof, we check that each member $\langle \varphi, a \rangle$ of the proof is $\mathcal{R}_{\mathrm{A}}$-provable in degree $b \in L$, $b \geq a$. So doing, we restrict ourselves only to proof members derived by (Ref)–(Rep).

(Ref): Take a term $t \in T(X)$. The sequence

1:	$\langle \emptyset \Rightarrow (t \approx t), 1 \rangle$,	axiom (ARef)
2:	$\langle P \Rightarrow (t \approx t), 1 \rangle$,	by (Mon) on 1

is an $\mathcal{R}_{\mathrm{A}}$-proof of $\langle P \Rightarrow (t \approx t), 1 \rangle$.

(Sym): Let $\langle P \Rightarrow (t' \approx t), a \rangle$ result from $\langle P \Rightarrow (t \approx t'), a \rangle$ by (Sym). Assuming $\Gamma \vdash^{\mathcal{R}_{\mathrm{A}}} \langle P \Rightarrow (t \approx t'), a' \rangle$ with $a \leq a'$, the sequence

	$\delta_1, \dots, \delta_l$,	
1:	$\langle P \Rightarrow (t \approx t'), a' \rangle$,	proof of $\langle P \Rightarrow (t \approx t'), a' \rangle$
2:	$\langle \langle t \approx t', a' \rangle \Rightarrow (t' \approx t), a' \rangle$,	axiom (ASym)
3:	$\langle P \Rightarrow (t' \approx t), a' \rangle$,	by (Mon) on 1, 2

is an $\mathcal{R}_{\mathrm{A}}$-proof of $\langle P \Rightarrow (t' \approx t), a' \rangle$ with $a \leq a'$ (note that $a' = a' \otimes (a' \to a')^*$).

(Tra): Let $\langle P \Rightarrow (t \approx t''), a \otimes b \rangle$ result from $\langle P \Rightarrow (t \approx t'), a \rangle$ and $\langle P \Rightarrow (t' \approx t''), b \rangle$ by (Tra). By induction, $\Gamma \vdash^{\mathcal{R}_{\mathrm{A}}} \langle P \Rightarrow (t \approx t'), a' \rangle$ and $\Gamma \vdash^{\mathcal{R}_{\mathrm{A}}} \langle P \Rightarrow (t' \approx t''), b' \rangle$, where $a \leq a'$ and $b \leq b'$. Moreover, consider the sequence

	$\delta_1, \dots, \delta_l$,	
1:	$\langle P \Rightarrow (t \approx t'), a' \rangle$,	proof of $\langle P \Rightarrow (t \approx t'), a' \rangle$
	$\delta'_1, \dots, \delta'_{l'}$,	
2:	$\langle P \Rightarrow (t' \approx t''), b' \rangle$,	proof of $\langle P \Rightarrow (t' \approx t''), b' \rangle$
3:	$\langle \langle t \approx t', a' \rangle \mathbin{\overline{\wedge}} \langle t' \approx t'', b' \rangle \Rightarrow (t \approx t''), a' \otimes b' \rangle$	axiom (ATra)
4:	$\langle P \Rightarrow (t \approx t''), a' \otimes b' \rangle$,	by (Mon) on 1, 2, 3

where $a' \otimes b' = (a' \otimes b') \otimes (a' \to a')^* \otimes (b' \to b')^*$. The above sequence is an $\mathcal{R}_{\mathrm{A}}$-proof of $\langle P \Rightarrow (t \approx t''), a' \otimes b' \rangle$ such that $a \otimes b \leq a' \otimes b'$.

(Rep): The proof is fully analogous to that of (Sym). ∎

THEOREM 3.8

Let $\mathcal{R}_{\mathrm{AG}} = \{(\text{ARef})\text{–}(\text{ARep}), (\text{Cut}), (\text{Sub}), (\text{Sup}), (\text{Wea})\}$. Then $\mathcal{R}_{\mathrm{AG}} \equiv \mathcal{R}$.

PROOF. "$\mathcal{R}_{\mathrm{AG}} \leq \mathcal{R}$": Consequence of Lemma 3.2, Lemma 3.3, and Lemma 3.7.

"$\mathcal{R}_{\mathrm{AG}} \geq \mathcal{R}$": We check that each $P \Rightarrow (t \approx t')$ is $\mathcal{R}_{\mathrm{AG}}$-provable in degree at least $P(t, t')$. This is true since

1:	$\langle \langle t \approx t', P(t,t') \rangle \Rightarrow (t \approx t'), P(t,t') \rangle$,	axiom (ARep)
2:	$\langle P \Rightarrow (t \approx t'), P(t,t') \rangle$	by (Wea) on 1

is an $\mathcal{R}_{\mathrm{AG}}$-proof of $\langle P \Rightarrow (t \approx t'), P(t,t') \rangle$. The rest follows by Lemma 3.4, and Lemma 3.7. ∎

Let us stress that the $\mathbf{L}^*$-deductive system $\mathcal{R}_{\mathrm{AG}}$ is further used in [17] when considering the problem of graded equational provability from extended systems of equational deduction rules. In [17] we also develop the above-sketched idea of representing certain deduction rules by Horn clauses considered as axioms.

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References

[1] Baaz M.: Infinite-valued Gödel logics with 0-1 projections and relativizations. *GÖDEL '96 – Logical Foundations of Mathematics, Computer Sciences and Physics,* Lecture Notes in Logic vol. 6, Springer-Verlag 1996, 23–33.

[2] Bělohlávek R.: Fuzzy equational logic. *Arch. Math. Log.* **41**(2002), 83–90.

[3] Bělohlávek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Publishers, New York, 2002.

[4] Bělohlávek R., Funioková T., Vychodil V.: Fuzzy closure operators with truth stressers. *Logic J. of IGPL* (to appear).

[5] Bělohlávek R., Vychodil V.: Fuzzy Horn logic I: proof theory. *Arch. Math. Log.* (to appear).

[6] Bělohlávek R., Vychodil V.: Fuzzy Horn logic II: implicationally defined classes. *Arch. Math. Log.* (to appear).

[7] Goguen J. A.: L-fuzzy sets. *J. Math. Anal. Appl.* **18**(1967), 145–174.

[8] Goguen J. A.: The logic of inexact concepts. *Synthese* **18**(1968–9), 325–373.

[9] Gottwald S.: *A Treatise on Many-Valued Logics.* Research Studies Press, Baldock, Hertfordshire, England, 2001.

[10] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[11] Hájek P.: On very true. *Fuzzy Sets and Systems* **124**(2001), 329–333.

[12] Novák V., Perfilieva I., Močkoř J.: *Mathematical Principles of Fuzzy Logic.* Kluwer, Boston, 1999.

[13] Palasińska K., Pigozzi G.: Gentzen-style axiomatizations in equational logic. *Algebra Universalis* **34**(1995), 128–143.

[14] Pavelka J.: On fuzzy logic I, II, III. *Z. Math. Logik Grundlagen Math.* **25**(1979), 45–52, 119–134, 447–464.

[15] Selman A.: Completeness of calculii for axiomatically defined classes of algebras. *Algebra Universalis* **2**(1972), 20–32.

[16] Takeuti G., Titani S.: Globalization of intuitionistic set theory. *Annals of Pure and Applied Logic* **33**(1987), 195–211.

[17] Vychodil V.: Extended fuzzy equational logic. *J. Mult. Val. Log. Soft Comput.* (to appear).

[18] Wechler W.: *Universal Algebra for Computer Scientists.* Springer-Verlag, Berlin Heidelberg, 1992.