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TRUTH-DEPRESSING HEDGES AND BL-LOGIC

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ABSTRACT. We show a complete axiomatization of unary connectives interpreted by monotone and superdiagonal truth functions, so-called truth-depressing hedges. These connectives formalize linguistic hedges like "slightly true" and "more or less". We follow ideas of [11] and show that BL_{vt}-logic can be enriched by a unary connective for which we can establish strong completeness with respect to the desired interpretation.

1. Introduction

Problem setting. We propose an axiomatization of unary connectives like "slightly true" and "more or less true" which extends the propositional BL-logic. Our motivation is the following: in [11], the author introduces a complete axiomatization of a logic which extends BL-logic by a unary connective "*vt*" that can be interpreted as "very true". Such a connective is interpreted by particular subdiagonal and monotone truth functions (i.e., particular unary functions defined on structures of truth degrees). Subdiagonality means that, denoting the interpretation of "*vt*" by v, $v(a) \leq a$ for each truth degree a; monotonicity says that $a \leq b$ implies $v(a) \leq v(b)$. Since each interpretation v of *vt* is truth-stressing due to subdiagonality, v is called a truth-stressing hedge. A borderline case of all possible interpretations of "very true" on a structure of truth degrees seems to be a mapping v defined by

$$v(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a \neq 1. \end{cases} \tag{1}$$

v defined by (1) is called a *globalization* [16]. Globalization can be seen as an interpretation of a connective "absolutely/fully true". The important point to note here is that a logical connective interpreted by (1) is axiomatizable in case of linearly ordered structures of truth degrees [1, 9].

Our aim is to look at connectives which can be seen as dual to "very true". In particular, we will be interested in connectives interpreted by unary truth functions which are monotone and *superdiagonal* (i.e., *truth-depressing*). That is, for a truth function s, we will require $a \leq s(a)$ for each truth degree a, meaning: "if a is true then a is slightly true". Observe that intuitively the greatest possible interpretation of "slightly true" may be a function s where

$$s(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \neq 0. \end{cases} \tag{2}$$

Described verbally, s defined by (2) can be seen as an interpretation of a connective "being not fully false". In more detail, $s(0) = 0$ says that "falsity is not even slightly true", and $s(a) = 1$ ($a \neq 0$) says that "not fully false" is "slightly true". Notice

2000 *Mathematics Subject Classification.* 03B52.
Key words and phrases. Axiomatization, BL-logic, fuzzy logic, truth-depressing hedge.

that such an s behaves like "globalization flipped over". In our paper, we introduce an axiomatization of a connective "slightly true" which allows (2) to be its boundary interpretation which is in addition axiomatizable in linear structures of truth degrees (linearly ordered BL-algebras endowed with additional unary functions).

Related works. The following list or related works is probably not exhaustive.

Macnab [14] introduced modal operators on Heyting algebras. Quite recently, in [13] the authors showed analogous modal operators on MV-algebras. Both the papers are devoted to algebraic properties of modal operators (unary functions satisfying certain conditions) and are not interested in their axiomatization in any logic. Their modal operators are idempotent (i.e., $f(f(a)) = f(a)$ for each $a \in L$) which seems to be too restrictive. In [15], F. Montagna studied storage operators which are interpreted by unary truth functions sending each $a \in L$ to the greatest idempotent below a. The results in [15] have shown strong completeness of important logics extended by storage (including BL and MTL) with respect to the corresponding linear structures of truth degrees. Fuzzy logics extended by modalities are discussed in [8].

In [16], the authors introduced so-called globalization which can be seen as an interpretation of connective "fully true". Baaz [1] studied this connective calling it $\triangle$ (interpreted by $\triangle(1) = 1$ and $\triangle(a) = 0$ for $a < 1$) for Gödel logics with any infinite set of truth values contained in the real interval $[0, 1]$ and containing 0 and 1. He formulated axioms for $\triangle$ and proved completeness of Gödel logic with these additional axioms for his semantics. Also BL with these additional axioms is complete over BL-chains with globalization [9] but the axioms are not sound for arbitrary (non-linear) BL-algebras with globalization. They are sound and complete for the class of so-called $\mathrm{BL}_{\triangle}$-algebras [9]. It is worth to mention that [1] also describes a dual connective $\bigtriangledown$ interpreted by $\bigtriangledown(0) = 0$ and $\bigtriangledown(a) = 1$ for $a > 0$, which is our desired truth function (2), however, the author does not pay attention to $\bigtriangledown$ because in logics considered in [1], $\bigtriangledown$ with its underlying interpretation is definable by $\bigtriangledown\varphi \equiv \neg\neg\varphi$.

Hájek and Harmancová [10] adopted Yashin axioms [17] of the "strong future tense operator" in Gödel logic and obtained a complete axiomatization for logical connective "more or less". An interesting thing is that this axiomatization does not use any additional deduction rules. On the other hand, the authors pointed out that Yashin axioms cannot give a nontrivial interpretation (other than identity) of "more or less" in case of Łukasiewicz logic. A hedge which can be interpreted as "more or less" appears also in [12].

In [11], Hájek has introduced logic BL_{vt} which is a conservative extension of BL-logic including logical connective "very true". The language of BL_{vt} extends the language of BL by a new unary connective "vt"; BL_{vt} contains three new axioms and a new deduction rule of truth confirmation (necessitation). BL_{vt} is strong complete w.r.t. semantics given by (linear) BL-algebras extended by a unary function v interpreting "vt" (so-called BL_{vt}-algebras), see [11] for details.

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Our paper is organized as follows. Section 2 contains preliminaries. In Section 3, we present a complete axiomatization of "slightly true". Section 4 contains examples of truth-depressing hedges on structures of truth degrees which represent interpretations of "slightly true". In Section 5, we discuss further possible axiomatizations of "slightly true" and show their relationship. Finally, Section 6 contains examples of applications of truth-depressing hedges in fuzzy relational systems.

2. Preliminaries

We assume that reader is familiar with basic fuzzy logic (BL) and its three most important schematic extensions: Gödel logic (G), Łukasiewicz logic (Ł), and product logic (Π), see [9]. Recall that (propositional) BL_{vt}-logic, introduced in [11], consists of the axioms of BL-logic and the following additional axioms,

$$vt\varphi \to \varphi, \tag{3}$$

$$vt(\varphi \to \psi) \to (vt\varphi \to vt\psi), \tag{4}$$

$$vt(\varphi \vee \psi) \to (vt\varphi \vee vt\psi) \tag{5}$$

plus the following deduction rule of truth confirmation:

$$\frac{\varphi}{vt\varphi}. \tag{6}$$

A BL_{vt}-algebra is a BL-algebra $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, 0, 1\rangle$ extended by a unary function $v\colon L \to L$ satisfying, for all $a, b \in L$,

$$v(1) = 1, \tag{7}$$

$$v(a) \leq a, \tag{8}$$

$$v(a \Rightarrow b) \leq v(a) \Rightarrow v(b), \tag{9}$$

$$v(a \cup b) \leq v(a) \cup v(b). \tag{10}$$

A theory (over BL_{vt}) is any set of formulas (of BL_{vt}). Given a BL_{vt}-algebra $\mathbf{L}$, an $\mathbf{L}$-evaluation e (for $vt\varphi$, $e(vt\varphi) = v(e(\varphi))$ where v in the unary operation in $\mathbf{L}$ interpreting "vt") is called an $\mathbf{L}$-model of a theory T if $e(\varphi) = 1$ for each $\varphi \in T$.

The following assertion is proved in [11]:

Theorem 1 (see [11])**.** *Let T be a theory over BL_{vt}, φ a formula. The following are equivalent:*

(i) *T proves φ over BL_{vt}.*
(ii) *For each (linearly ordered) BL_{vt}-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T, $e(\varphi) = 1$ (φ is $\mathbf{L}$-true in e).* □

In what follows we use the notation of [9, 11].

3. Axiomatization of "slightly true"

In this section we show an axiomatization of "slightly true" which will be related to the axiomatization of "very true" [11]. The basic idea of our development is to have one axiomatization for both the connectives. A joint axiomatization enables us to use the existing results [11]. In addition to that, from the epistemic point of view, one may argue that "slightly" is usually, and maybe subconsciously, compared to "very". For instance, in many situations, it is natural to claim that "if something is slightly true, then it is not (very/fully) false". Thus, axiomatizing "slightly true" and "very true" together seems to be beneficial from both technical and epistemic points of view. We will be interested only in propositional case.

We now introduce a logic which extends BL_{vt} by a new unary connective "slightly true" denoted "st". Call this logic $\mathrm{BL}_{vt,st}$-logic. The language of $\mathrm{BL}_{vt,st}$-logic is an extension of the language of BL_{vt}; we extend the notion of a formula accordingly so that each $st\varphi$ is a formula of $\mathrm{BL}_{vt,st}$. $\mathrm{BL}_{vt,st}$ contains all axioms and deduction rules of BL_{vt} plus the following axioms describing properties of st:

$$\varphi \to st\varphi, \tag{11}$$

$$st\varphi \to \neg vt\neg\varphi, \tag{12}$$

$$vt(\varphi \to \psi) \to (st\varphi \to st\psi). \tag{13}$$

Axiom (11) is quite expected as says that "everything true is slightly true"; axiom (12) is a formalization of the above-mentioned relationship between "very" and "slightly": "if φ is slightly true, then φ is not very false"; axiom (13) is a type of transitivity of "slightly true". Notice that on the left-hand side of (13), there is "vt", however, on the right-hand side of it we have two occurrences of "st". Roughly speaking, the axiom says that "if φ is slightly true and $\varphi \to \psi$ is very true, then ψ is slightly true". Thus, the deduction rule *modus ponens* (MP) allows us to infer "slightly true consequent from very true implication with slightly true antecedent" which seems to be natural.

Remark 1. For our purposes, axiom (13) is more acceptable than

$$st(\varphi \to \psi) \to (st\varphi \to st\psi) \tag{14}$$

which has been used, e.g., in [10]. The basic problem with (14) is that postulating (14) would rule out important interpretations of "st". For instance, if $\mathbf{L}$ is the standard Łukasiewicz algebra and $s\colon [0,1] \to [0,1]$ is defined by (2), then $\mathbf{L}$ endowed with s does not satisfy $s(a \Rightarrow b) \leq s(a) \Rightarrow s(b)$: for $a = 0.1$ and $b = 0$ we have $s(a \Rightarrow b) = s(0.9) = 1 \nleq 0 = 1 \Rightarrow 0 = s(a) \Rightarrow s(b)$. Therefore, axiom (14) is not sound for the standard Łukasiewicz algebra with s defined by (2).

The following assertion shows that axiom (12) can be equivalently replaced by axiom $\neg st\overline{0}$, saying "falsity is not (even) slightly true".

Lemma 2. *In $BL_{vt,st}$, axiom* (12) *can be equivalently replaced by axiom $\neg st\overline{0}$.*

Proof. "$\Rightarrow$": We show that $\neg st\overline{0}$ is provable in $\mathrm{BL}_{vt,st}$.
$\vdash st\overline{0} \to \neg vt\neg\overline{0}$ [axiom (12)]
$\vdash \neg\neg vt\neg\overline{0} \to \neg st\overline{0}$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$ and MP]
$\vdash vt\neg\overline{0} \to \neg st\overline{0}$ [by $\chi \to \neg\neg\chi$ and transitivity of implication]
$\vdash \neg\overline{0}$ [axiom of BL]
$\vdash vt\neg\overline{0}$ [by (6)]
$\vdash \neg st\overline{0}$ [by MP]
"$\Leftarrow$": It suffices to check that BL_{vt} plus (11), (13), and $\neg st\overline{0}$ prove (12).
$\vdash vt(\varphi \to \overline{0}) \to (st\varphi \to st\overline{0})$ [axiom (13)]
$\vdash (st\varphi \,\&\, vt(\varphi \to \overline{0})) \to st\overline{0}$ [using $(\chi \to (\vartheta \to \gamma)) \equiv ((\vartheta \,\&\, \chi) \to \gamma)$]
$\vdash st\overline{0} \to \overline{0}$ [assumption $\neg st\overline{0}$]
$\vdash (st\varphi \,\&\, vt(\varphi \to \overline{0})) \to \overline{0}$ [by transitivity of implication]
$\vdash st\varphi \to (vt(\varphi \to \overline{0}) \to \overline{0})$ [by $((\chi \,\&\, \vartheta) \to \gamma) \to (\chi \to (\vartheta \to \gamma))$ and MP]
$\vdash st\varphi \to \neg vt\neg\varphi$ [shorthand for the latter formula] □

Remark 2. Note that one may expect that we introduce a deduction rule

$$\frac{\neg\varphi}{\neg st\varphi}, \tag{15}$$

which can be seen as a rule dual to (6). This is not necessary because if T proves $\neg\varphi$ over $\mathrm{BL}_{vt,st}$, then, by (6) we get that T proves $vt(\varphi \to \overline{0})$ over $\mathrm{BL}_{vt,st}$, i.e. using (13) and modus ponens, it follows that T proves $st\varphi \to st\overline{0}$ over $\mathrm{BL}_{vt,st}$. Hence, transitivity of implication together with Lemma 2 yield that T proves $\neg st\varphi$ over $\mathrm{BL}_{vt,st}$. Hence, (15) is a rule which is derivable in $\mathrm{BL}_{vt,st}$. Nevertheless, we are going to use (15) in Section 5.

Before we introduce the interpretation of formulas of $\mathrm{BL}_{vt,st}$-logic, let us show that two important fixed connectives "*st*" are definable inside $\mathrm{BL}_{vt,st}$. We will take advantage of the following lemma.

Lemma 3. *BL_{vt} proves the following formulas,*

(i) $\varphi \to \neg vt\neg\varphi$,
(ii) $vt(\varphi \to \psi) \to (\neg vt\neg\varphi \to \neg vt\neg\psi)$.

Proof.
"(i)":

$\vdash vt\neg\varphi \to \neg\varphi$	[axiom (3)]
$\vdash \neg\neg\varphi \to \neg vt\neg\varphi$	[by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$ and MP]
$\vdash \varphi \to \neg vt\neg\varphi$	[by $\chi \to \neg\neg\chi$ and MP]

"(ii)":

$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	[axiom of BL]
$\vdash vt((\varphi \to \psi) \to (\neg\psi \to \neg\varphi))$	[by (6)]
$\vdash vt(\varphi \to \psi) \to vt(\neg\psi \to \neg\varphi)$	[by (4) and MP]
$\vdash vt(\neg\psi \to \neg\varphi) \to (vt\neg\psi \to vt\neg\varphi)$	[axiom (4)]
$\vdash vt(\varphi \to \psi) \to (vt\neg\psi \to vt\neg\varphi)$	[by transitivity of implication]
$\vdash (vt\neg\psi \to vt\neg\varphi) \to (\neg vt\neg\varphi \to \neg vt\neg\psi)$	[axiom of BL]
$\vdash vt(\varphi \to \psi) \to (\neg vt\neg\varphi \to \neg vt\neg\psi)$	[by transitivity of implication] □

Remark 3. Applying Lemma 3, if we let $st\varphi$ be a shorthand for $\neg vt\neg\varphi$, then (11)–(13) would be provable in BL_{vt}. Indeed, (11) would be covered by Lemma 3 (i), (12) would be of the form $\vartheta \to \vartheta$ which is provable in BL, and (13) would be provable on account of Lemma 3 (ii). That is, $\neg vt\neg\varphi$ is a particular $st\varphi$ which is definable inside $\mathrm{BL}_{vt,st}$ by postulating $\neg vt\neg\varphi \to st\varphi$ (notice that the converse implication $st\varphi \to \neg vt\neg\varphi$ is one of the axioms of $\mathrm{BL}_{vt,st}$). Analogously, if $st\varphi$ is a shorthand for φ then (11)–(13) are also provable in BL_{vt} because (11) becomes $\varphi \to \varphi$, (12) is provable due to Lemma 3 (i), and (13) becomes an instance of (3). As a consequence, this particular "*st*" is definable inside $\mathrm{BL}_{vt,st}$ by $st\varphi \to \varphi$ (again, the converse implication is an axiom of $\mathrm{BL}_{vt,st}$). In fact, these two definitions of "*st*" ($st\varphi$ is φ, and $st\varphi$ is $\neg vt\neg\varphi$) are two borderline cases of connectives associated with "*vt*". We will comment of this later on.

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In order to interpret formulas of $\mathrm{BL}_{vt,st}$, we extend BL_{vt}-algebras by an additional unary operation: a *$BL_{vt,st}$-algebra* $\mathbf{L}$ is a BL_{vt}-algebra $\langle L, \cup, \cap, *, \Rightarrow, v, 0, 1\rangle$

endowed with a unary operation $s\colon L \to L$ satisfying, for all $a, b \in L$,

$$s(0) = 0, \tag{16}$$

$$a \leq s(a), \tag{17}$$

$$v(a \Rightarrow b) \leq s(a) \Rightarrow s(b); \tag{18}$$

s will be called a *truth-depressing hedge* (*associated with* v). Given a $\mathrm{BL}_{vt,st}$-algebra $\mathbf{L}$ and an $\mathbf{L}$-evaluation e we put $e(st\varphi) = s(e(\varphi))$; e is a model of a theory T (over $\mathrm{BL}_{vt,st}$) if, for each $\varphi \in T$, $e(\varphi) = 1$. Now, we have

Lemma 4. *Let* $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, v, s, 0, 1\rangle$ *be a* $BL_{vt,st}$*-algebra,* $\mathbf{L}' = \langle L, \cup, \cap, *, \Rightarrow, v, 0, 1\rangle$ *be a* BL_{vt}*-algebra which is a reduct of* $\mathbf{L}$*. Let* $s_{\min}\colon L \to L$*,* $s_{\max}\colon L \to L$ *be mappings defined by*

$$s_{\min}(a) = a, \tag{19}$$

$$s_{\max}(a) = v(a \Rightarrow 0) \Rightarrow 0, \tag{20}$$

for each $a \in L$*. Then*

(i) *s is monotone, i.e.* $a \leq b$ *implies* $s(a) \leq s(b)$*,*
(ii) $s_{\min}(a) \leq s(a) \leq s_{\max}(a)$ *for each* $a \in L$*,*
(iii) $\mathbf{L}'$ *endowed with* $s_{\min}$ *is a* $BL_{vt,st}$*-algebra,*
(iv) $\mathbf{L}'$ *endowed with* $s_{\max}$ *is a* $BL_{vt,st}$*-algebra,*
(v) $BL_{vt,st}$ *is sound for* $\mathbf{L}$*, i.e. if* T *proves* φ *over* $BL_{vt,st}$*, then* $e(\varphi) = 1$ *for each* $\mathbf{L}$*-model* e *of* T*.*

Proof. "(i)": If $a \leq b$, (7) and (18) give $1 = v(1) = v(a \Rightarrow b) \leq s(a) \Rightarrow s(b)$, i.e. $s(a) \leq s(b)$.

"(ii)": $s_{\min}(a) = a \leq s(a)$ due to (17). Using adjointness, $s(a) \leq (s(a) \Rightarrow 0) \Rightarrow 0$, i.e. $s(a) \leq (s(a) \Rightarrow s(0)) \Rightarrow 0$ by (16), from which we get $s(a) \leq v(a \Rightarrow 0) \Rightarrow 0$ by (18) and antitony of $\Rightarrow$ in its first argument. Thus, $s(a) \leq s_{\max}(a)$.

"(iii)": $s_{\min}$ trivially satisfies (16) and (17). (18) is satisfied due to (8).

"(iv)": We have $s_{\max}(0) = v(0 \Rightarrow 0) \Rightarrow 0 = v(1) \Rightarrow 0 = 1 \Rightarrow 0 = 0$, i.e. $s_{\max}$ satisfies (16). Due to (8), we have $v(a \Rightarrow 0) \leq a \Rightarrow 0$. Thus, adjointness yields $a \leq v(a \Rightarrow 0) \Rightarrow 0 = s_{\max}(a)$, i.e. $s_{\max}$ satisfies (17). Finally, we show (18): from transitivity of residuum we get $(a \Rightarrow b) * (b \Rightarrow 0) \leq a \Rightarrow 0$, i.e. $a \Rightarrow b \leq (b \Rightarrow 0) \Rightarrow (a \Rightarrow 0)$, by monotony of v: $v(a \Rightarrow b) \leq v((b \Rightarrow 0) \Rightarrow (a \Rightarrow 0))$, and $v(a \Rightarrow b) \leq v(b \Rightarrow 0) \Rightarrow v(a \Rightarrow 0)$, this further gives $v(a \Rightarrow b) \leq (v(a \Rightarrow 0) \Rightarrow 0) \Rightarrow (v(b \Rightarrow 0) \Rightarrow 0) = s_{\max}(a) \Rightarrow s_{\max}(b)$, proving the claim.

"(v)": Let T be a theory, e be an $\mathbf{L}$-model of T. For each instance φ of (11)–(13) we have $e(\varphi) = 1$. Indeed, in case of (11) and (13), this is a consequence of (17) and (18); for (12), apply Lemma 4 (ii). The rest is done by induction on the length of a proof. □

Remark 4. Using Lemma 4 (ii)–(iv), we get the following consequence. Each BL_{vt}-algebra $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, v, 0, 1\rangle$ can be extended to a $\mathrm{BL}_{vt,st}$-algebra by adding a unary function $s : L \to L$ defined by (19) or (20). Moreover, (19) is the least possible truth-depressing hedge associated with v while (20) is the greatest one. This observation is a semantic counterpart of the conclusion of Remark 3.

Finally, we prove that $\mathrm{BL}_{vt,st}$ is strong complete with respect to semantics given by $\mathrm{BL}_{vt,st}$-algebras.

Lemma 5. *Let T be a theory over $BL_{vt,st}$. If T does not prove φ over $BL_{vt,st}$, then there is a complete theory $T' \supseteq T$ such that T' does not prove φ over $BL_{vt,st}$.*

Proof. Since $\mathrm{BL}_{vt,st}$ results from BL_{vt} by extending the language and adding new axioms but *without* introducing any new deduction rules, Lemma 5 can be proved analogously as in case of BL_{vt} [11]. Therefore, we present only a sketch of the proof. First, one can show that $\mathrm{BL}_{vt,st}$ has a *deduction theorem* of the following form: $T \cup \{\varphi\}$ proves ψ over $\mathrm{BL}_{vt,st}$ iff there is n such that T proves $\tau^n\varphi \to \psi$ over $\mathrm{BL}_{vt,st}$, where $\tau^n\varphi$ [11] denotes the formula defined by $\tau^0\varphi = \varphi$, $\tau^{i+1}\varphi = vt(\tau^i\varphi \mathbin{\&} \tau^i\varphi)$. This can be shown using the same arguments as in [11] (recall that $\mathrm{BL}_{vt,st}$ has the same deduction rules as BL_{vt}). Then, using the deduction theorem, Lemma 5 follows almost immediately (check the corresponding claim from [11]). □

Theorem 6 (strong completeness of $\mathrm{BL}_{vt,st}$)**.** *Let T be a theory over $BL_{vt,st}$, φ a formula. The following are equivalent:*

(i) *T proves φ over $BL_{vt,st}$.*
(ii) *For each (linearly ordered) $BL_{vt,st}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T, $e(\varphi) = 1$.*

Proof. Given any T, one can define a Lindenbaum algebra [9] $\mathbf{L}_T$ of T-equivalent formulas which is a well-defined $\mathrm{BL}_{vt,st}$-algebra. This is clear because if T proves $\varphi \to \psi$ and $\psi \to \varphi$ over $\mathrm{BL}_{vt,st}$ then, by (6), T proves $vt(\varphi \to \psi)$ and $vt(\psi \to \varphi)$, i.e., by (13) and modus ponens, T proves $st\varphi \to st\psi$ and $st\psi \to st\varphi$, which means $[st\varphi]_T = [st\psi]_T$, i.e. $\mathbf{L}_T$ is well defined, and $s([\varphi]_T) = [st\varphi]_T$ satisfies (16)–(18). The latter is easy to see. For instance, in order to check (16) it suffices to show that both $\overline{0} \to st\overline{0}$ and $st\overline{0} \to \overline{0}$ are provable in $\mathrm{BL}_{vt,st}$, which is indeed true (the first formula is an axiom of BL, provability of the second one follows from Lemma 2). If T does not prove φ, then due to Lemma 5 there is a complete $T' \supseteq T$ which does not prove φ. Since T' is complete, $\mathbf{L}_{T'}$ is a linearly ordered $\mathrm{BL}_{vt,st}$-algebra. For an $\mathbf{L}_{T'}$-model e of T' (which is also an $\mathbf{L}_{T'}$-model of T), defined by $e(\psi) = [\psi]_{T'}$ (for each ψ), we have $e(\varphi) \neq [\,\overline{1}\,]_{T'} = 1$. The rest follows from Lemma 4 (v). □

We conclude this section by three remarks.

Remark 5. Using Theorem 6, we immediately get that $st(\varphi \wedge \psi) \equiv (st\varphi \wedge st\psi)$ and $st(\varphi \vee \psi) \equiv (st\varphi \vee st\psi)$ are provable in $\mathrm{BL}_{vt,st}$. Indeed, this follows from the fact that both the formulas are tautologies in each linearly ordered $\mathrm{BL}_{vt,st}$-algebra.

Remark 6. As in [11], we get that $\mathrm{BL}_{vt,st}$ is a conservative extension of BL_{vt}. In more detail, we claim that if T is a theory over BL_{vt} and T proves φ over $\mathrm{BL}_{vt,st}$ then T proves φ over BL_{vt}. This is almost evident because if T does not prove φ over BL_{vt} then, by completeness of BL_{vt}, there is a linear $\mathbf{L}$-model e of T such that $e(\varphi) \neq 1$; furthermore $\mathbf{L}$ can be extended to a $\mathrm{BL}_{vt,st}$-algebra by s defined by (19), i.e. this way we obtain a linear $\mathrm{BL}_{vt,st}$ algebra $\mathbf{L}'$ such that φ is not $\mathbf{L}'$-true in e; by soundness of $\mathrm{BL}_{vt,st}$, T does not prove φ. Furthermore, since BL_{vt} is a conservative extension of BL [11], we get that $\mathrm{BL}_{vt,st}$ is a conservative extension of BL.

Remark 7. We can get, of course, strong completeness for schematic extensions of $\mathrm{BL}_{vt,st}$. This follows the same ideas as in [9, 11]. In particular, we obtain strong completeness of logics $\mathrm{G}_{vt,st}$, $\mathrm{Ł}_{vt,st}$, and $\Pi_{vt,st}$ over (linearly ordered) $\mathrm{G}_{vt,st}$-algebras, $\mathrm{MV}_{vt,st}$-algebras, and $\Pi_{vt,st}$-algebras, respectively. Truth-depressing hedges in context of these stronger logics will be discussed later on.

4. Examples

In this section we show several examples of truth-depressing hedges associated with truth-stressing ones. We focus mainly on hedges defined on linearly ordered BL-algebras. Let us note that if we are given a BL_{vt}-algebra $\mathbf{L}$, then v (truth-stressing hedge on $\mathbf{L}$) can be seen as a constraint for possible choices of a truth-depressing hedge s associated with v because s has to satisfy condition (18) which is parameterized by v. We have already shown that for each v there are always at least two boundary choices of s (which may coincide), see Lemma 4. In general, for each v defined on $\mathbf{L}$ there is a whole family of associated truth-depressing hedges. Some of the subsequent examples will show how the choice of a truth-stressing hedge v affect the family of associated truth-depressing hedges.

For brevity, the standard Łukasiewicz, Gödel, and product (Goguen) algebras will be denoted by $[0,1]_{\mathrm{Ł}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$, respectively.

Example 1. Suppose $\mathbf{L}$ is a linear BL-algebra. $\mathbf{L}$ equipped with v defined by (1) is a BL_{vt}-algebra [11]. In this case, each monotone $s\colon L \to L$ satisfying (16) and (17) is a truth-depressing hedge associated with v. Indeed, (18) is satisfied because we either have $a \leq b$ and thus $s(a) \leq s(b)$ due to the monotony of s which yields $v(a \Rightarrow b) \leq 1 = s(a) \Rightarrow s(b)$, or $a \nleq b$ and thus $v(a \Rightarrow b) = 0 \leq s(a) \Rightarrow s(b)$. Therefore, the family of associated truth-depressing hedges contains any monotone superdiagonal truth function which sends zero to zero. Note also that globalization on general (nonlinear) $\mathbf{L}$ need not satisfy condition (10) (consider, e.g., globalization on four element Boolean algebra).

Example 2. If $\mathbf{L}$ is a linear Gödel algebra, then $v : L \to L$ is a truth-stressing hedge on $\mathbf{L}$ iff v is monotone, subdiagonal, and satisfies (7), see [11]. Indeed, if $a \leq b$ then, due to monotony, $v(a \Rightarrow b) \leq 1 = v(a) \Rightarrow v(b)$. If $a \nleq b$ we either have $v(a) = v(b)$ in which case (9) is trivially satisfied, or $v(a) \nleq v(b)$, i.e. $v(a) \Rightarrow v(b) = v(b)$ due to properties of residuum in Gödel chains, thus, $v(a \Rightarrow b) = v(b) = v(a) \Rightarrow v(b)$. So, each monotone and subdiagonal v satisfying $v(1) = 1$ is a truth-stressing hedge on $\mathbf{L}$. Now, in an analogous way, each monotone and superdiagonal truth function $s : L \to L$ satisfying (16) is a truth-depressing hedge (associated with any v). Indeed, for any v, $a \nleq b$ and $s(a) \nleq s(b)$ imply $v(a \Rightarrow b) = v(b) \leq b \leq s(b) = s(a) \Rightarrow s(b)$. The rest is obvious. So, from the point of view of truth-stressing/depressing hedges, linear Gödel algebras are among all the (linear) BL-algebras the least restrictive ones.

Example 3. As shown in [5], for truth degrees $c_1, \dots, c_k \in L$ and nonnegative integers $n_1, \dots, n_k$, $v\colon L \to L$ defined by

$$v(a) = \begin{cases} 1 & \text{if } a = 1, \\ \bigcup_{i=1}^{k} (c_i * a^{n_i}) & \text{if } a < 1, \end{cases} \tag{21}$$

satisfies (7), (8), and (9). Thus, each linearly ordered BL-algebra equipped with v defined by (21) is a BL_{vt}-algebra. The assumption of linearity is essential because otherwise (10) would not be satisfied in general. By Lemma 4, the greatest truth-depressing hedge on such BL_{vt}-algebra (i.e., associated with v) is

$$s(a) = \begin{cases} 0 & \text{if } a = 0, \\ \bigcap_{i=1}^{k} ((c_i * (a \Rightarrow 0)^{n_i}) \Rightarrow 0) & \text{if } a > 0. \end{cases} \tag{22}$$

Definitions (21) and (22) coincide with (1) and (2) for $c_1 = \cdots = c_k = 0$.

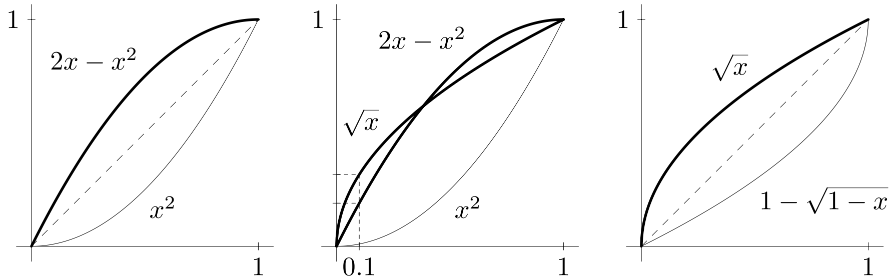

FIGURE 1. Truth-depressing hedges on the standard Łukasiewicz algebra

Example 4. A particular case of Example 3 is that of $v(a) = a * a$ (i.e., we consider just one truth degree $c_1 = 1$ and put $n_1 = 2$). In this case, the greatest truth-depressing hedge associated with v satisfies $s(a) = ((a \Rightarrow 0) * (a \Rightarrow 0)) \Rightarrow 0$. Observe the behavior of $v(a) = a * a$ and s in standard Łukasiewicz, product and Gödel algebras. For $\mathbf{L}$ being $[0,1]_{\mathrm{L}}$, $v(a) = \max(2a-1, 0)$, $s(a) = \min(2a, 1)$. As a consequence, s' defined by (2) is not a truth-depressing hedge associated with v because $s'(0.1) = 1 \nleq 0.2 = \min(2 \cdot 0.1, 1) = s(0.1)$. On the other hand, if $\mathbf{L}$ is $[0,1]_{\Pi}$, then $v(a)$ is the algebraic power and $s(a)$ coincides with (2); if $\mathbf{L}$ is $[0,1]_{\mathrm{G}}$, v is identity and s also coincides with (2).

Example 5. For $v(a) = a^2$ (algebraic power), v is a truth-stressing hedge on $[0,1]_{\Pi}$ (this is covered by the previous example), $[0,1]_{\mathrm{L}}$, and $[0,1]_{\mathrm{G}}$, see [11]. Algebraic power is a popular choice of truth-stressing (linguistic) hedges in applications. Analogously, as in Example 4, for $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$, s defined by (2) is the greatest truth-depressing hedge associated with v. In case of $[0,1]_{\mathrm{L}}$, the greatest associated truth-depressing hedge is given by $s(a) = 2a - a^2$, see Fig. 1 (left).

Example 6. In applications, the square root is traditionally being used as a truth-depressing (linguistic) hedge (usually together with x^2). Thus, given a BL-algebra on $[0,1]$, we should ask for what truth-stressing hedges (if any), s given by $s(a) = \sqrt{a}$ is an associated truth-depressing hedge. From Example 1 it follows that any BL-algebra on $[0,1]$ (with its genuine ordering) with globalization and s being $\sqrt{x}$ is a $\mathrm{BL}_{vt,st}$-algebra. Moreover, $[0,1]_{\mathrm{G}}$ with any v, and s being $\sqrt{x}$, is a $\mathrm{BL}_{vt,st}$-algebra, see Example 2. This applies also for $[0,1]_{\Pi}$ because for $a \nleq b$ we have

$$a \Rightarrow b = \frac{b}{a} \leq \sqrt{\tfrac{b}{a}} = \frac{\sqrt{b}}{\sqrt{a}} = \sqrt{a} \Rightarrow \sqrt{b}.$$

The latter inequality yields that $[0,1]_{\Pi}$ with v being identity and s being $\sqrt{x}$ is a $\mathrm{BL}_{vt,st}$-algebra. Since identity is the greatest truth-stressing hedge, $[0,1]_{\Pi}$ with any v and $\sqrt{x}$ is a $\mathrm{BL}_{vt,st}$-algebra. Contrary to that, not all truth-stressing hedges on $[0,1]_{\mathrm{L}}$ allow $\sqrt{x}$ to be an associated truth-depressing hedge. This is the case of, e.g., v given by x^2. In more detail, we have $\sqrt{a} \nleq 2a - a^2$ for $a = 0.1$, see Fig. 1 (middle), i.e. $\sqrt{x}$ is not below the greatest truth-depressing hedge associated with x^2, cf. Example 5. Therefore, $\sqrt{x}$ is not associated with x^2 in $[0,1]_{\mathrm{L}}$.

Example 7. Take $[0,1]_{\mathrm{L}}$ and define v by $v(a) = 1 - \sqrt{1-a}$. Obviously, v satisfies (7), (8), and (10) (the latter is satisfied due to linearity). It also satisfies (9): if $a \leq b$, (9) holds due to monotony of v; if $a \nleq b$ and $v(a) \nleq v(b)$, which is the

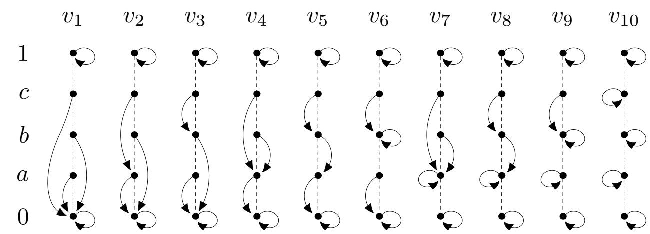

FIGURE 2. Truth-stressing hedges on five-element Łukasiewicz chain

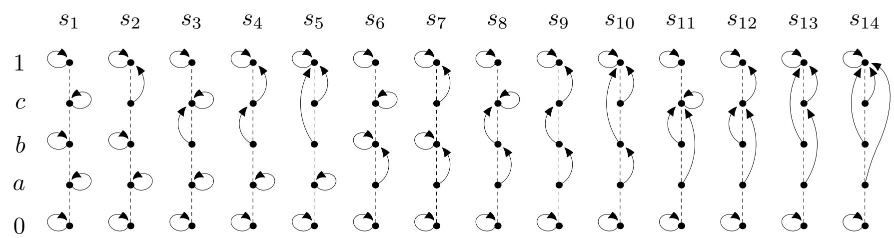

FIGURE 3. Monotone superdiagonal truth functions sending 0 to 0

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}
v_1	×	×	×	×	×	×	×	×	×	×	×	×	×	×
v_2	×	×	×	×	×	×	×	×	×	×	×	×	×	
v_3	×	×	×	×		×	×	×	×	×				
v_4	×	×	×	×		×	×	×	×		×	×		
v_5	×	×	×	×		×	×	×	×					
v_6	×					×	×							
v_7	×		×			×		×			×			
v_8	×		×			×		×						
v_9	×					×								
v_{10}	×													

FIGURE 4. Truth-stressing and truth-depressing hedges

nontrivial case, $v(a \Rightarrow b) = 1 - \sqrt{a-b}$ and $v(a) \Rightarrow v(b) = \sqrt{1-a} + 1 - \sqrt{1-b}$, i.e. it suffices to show $1 - \sqrt{a-b} \leq \sqrt{1-a} + 1 - \sqrt{1-b}$, i.e. it suffices to show $\sqrt{1-b} - \sqrt{1-a} \leq \sqrt{(1-b)-(1-a)}$. Since $\sqrt{1-b}$, $\sqrt{1-a}$, and $\sqrt{(1-b)-(1-a)}$ form a rectangular triangle, the desired inequality follows from triangular inequality. Altogether, $[0,1]_{\text{Ł}}$ with v is a BL_{vt}-algebra. Applying Lemma 4 (ii) (iv), s defined by $s(a) = v(a \Rightarrow 0) \Rightarrow 0 = 1 - v(1-a) = 1 - (1 - \sqrt{1-(1-a)}) = \sqrt{a}$ is the greatest truth-depressing hedge associated with v, i.e. $[0,1]_{\text{Ł}}$ equipped with such v and s is a $\text{BL}_{vt,st}$-algebra, see Fig. 1 (right).

Example 8. Consider a five-element Łukasiewicz chain where $0 < a < b < c < 1$. Fig. 2 depicts all truth-stressing hedges on such a BL-algebra: the left-most one is globalization, the right-most one is identity. Fig. 3 depicts all monotone superdiagonal truth functions satisfying (16): the left-most one is identity, the right-most one is given by (2). Table in Fig.4 displays relationship between truth-stressing hedges from Fig. 2 and truth functions from Fig. 3. Each entry "×" in the table indicates that the truth function given by column is a truth-depressing hedge associated with the truth-stressing hedge given by row. In general, if v is the identity mapping on a Łukasiewicz chain, then the only truth-depressing hedge associated with v is the identical mapping (see row v_{10} in Fig. 4). By Example 1, if v is globalization on Łukasiewicz chain then any monotone and superdiagonal truth function satisfying (16) is an associated truth-depressing hedge (see row v_1 in Fig. 4).

Example 9. In previous examples, we have seen that Gödel chains do not restrict truth-stressing or truth-depressing hedges. In other words, monotone unary functions which are subdiagonal/superdiagonal and satisfy (7)/(16) are identified with truth-stressing/truth-depressing hedges (for any v). We have also demonstrated that in case of $[0,1]_{Ł}$, a definition of v has essential influence on the family of associated truth-depressing hedges. The same applies for $[0,1]_{\Pi}$. For illustration, let v be the identity mapping, and define s by

$$s(a) = \begin{cases} a & \text{if } a < 0.5, \\ 1 & \text{else.} \end{cases}$$

Clearly, s is monotone, superdiagonal, and (16) is satisfied, however, s is not a truth-depressing hedge associated with v because $v(0.5 \Rightarrow 0.4) = v(0.8) = 0.8 \nleq 0.4 = s(0.4) = 1 \Rightarrow s(0.4) = s(0.5) \Rightarrow s(0.4)$, showing that (18) does not hold.

5. More on axiomatizations of "slightly true"

In this section we present axiomatizations of "slightly true" which are not based on BL_{vt}. We focus mainly on introducing "slightly true" in three important extensions of BL-logic: G (Gödel), Ł (Łukasiewicz), and Π (product) logics. Our basic requirement is to have axiomatizations of monotone superdiagonal truth functions which allow (2) to be the borderline interpretation of "slightly true".

A straightforward way to go is the following. We use a modified version of axioms (11)–(13): (11) does not contain "*vt*" so we can adopt it "as is"; in $\mathrm{BL}_{vt,st}$, (12) is equivalent to $\neg st\overline{0}$ (see Lemma 2), i.e. we use $\neg st\overline{0}$ instead of (12); we remove "*vt*" from (13), or we can replace it by "*st*". This way we arrive to two extensions of BL with working names (I) and (II):

(I) Extend BL-logic by connective "*st*", and axioms

$$\neg st\overline{0}, \tag{23}$$

$$\varphi \to st\varphi, \tag{24}$$

$$st(\varphi \to \psi) \to (st\varphi \to st\psi). \tag{25}$$

(II) Proceed the same way as in (I) only replace (25) by

$$(\varphi \to \psi) \to (st\varphi \to st\psi). \tag{26}$$

Both (I) and (II) are strong complete over semantics given by (linearly ordered) BL-algebras endowed with a unary operation s satisfying, $s(0) = 0$, $a \leq s(a)$, and $s(a \Rightarrow b) \leq s(a) \Rightarrow s(b)$ or $a \Rightarrow b \leq s(a) \Rightarrow s(b)$, respectively. This is easy to see because, in both cases, we can use the fact that each consistent theory has a consistent completion and the standard construction of Lindenbaum algebras [9] (with s being their fundamental operation): in case of (I) such an algebra is well defined because if T proves $\varphi \to \psi$, then T proves $st(\varphi \to \psi)$ (use (24) and MP), and consequently T proves $st\varphi \to st\psi$ (by (25) and MP); in case of (II), if T proves $\varphi \to \psi$, then T proves $st\varphi \to st\psi$ (by (26) and MP).

Remark 8. In case of Gödel and product logics, s defined by (2) is a sound interpretation of (I) and (II) over linearly ordered Gödel and product algebras. Indeed, $s(a) \Rightarrow s(b)$ either is 0 or 1. If $s(a) \Rightarrow s(b) = 0$ (nontrivial case), then $a \neq 0$ and $b = 0$, i.e. we get $a \Rightarrow 0 = 0$ (recall that in linear Gödel and product algebras, either a or $a \Rightarrow 0$ is 0) and $b = 0$, that is, $0 = a \Rightarrow 0 = a \Rightarrow b$ yielding $s(a \Rightarrow b) = 0$. Therefore, for each truth degrees a, b, we have $a \Rightarrow b \leq s(a \Rightarrow b) \leq s(a) \Rightarrow s(b)$.

Remark 9. In both (I) and (II), axiom (23) can be equivalently replaced by $st\varphi \to \neg\neg\varphi$. In more detail, by $\vdash st\overline{0} \to \neg\neg\overline{0}$ and $\vdash \neg\neg\overline{0} \to \overline{0}$ we get $\vdash st\overline{0} \to \overline{0}$, i.e. $\vdash \neg st\overline{0}$; conversely, in case of (I), $\vdash \neg st\overline{0}$ and $\vdash st(\varphi \to \overline{0}) \to (st\varphi \to st\overline{0})$ give $\vdash (st\varphi \mathbin{\&} st(\varphi \to \overline{0})) \to st\overline{0})$, and thus $\vdash (st\varphi \mathbin{\&} st(\varphi \to \overline{0})) \to \overline{0})$, which further gives $\vdash st(\varphi \to \overline{0}) \to (st\varphi \to \overline{0})$, $\vdash (\varphi \to \overline{0}) \to (st\varphi \to \overline{0})$, and $\vdash st\varphi \to ((\varphi \to \overline{0}) \to \overline{0})$, i.e. $\vdash st\varphi \to \neg\neg\varphi$. The case of (II) is fully analogous. Our observation has the following consequence: if we introduce (I) in Gödel logic and add axiom $ststst\varphi \to st\varphi$, which together with (24) say that "*st* is idempotent", we obtain a proof system which is equivalent to that one from [10].

Observe that neither of (I) and (II) allows s defined by (2) to be a sound interpretation of "*st*" in $[0,1]_{\text{Ł}}$. Thus, (I) and (II) are not sound over linearly ordered MV-algebras. In the rest of this section we show an extension of Ł which is not based on (I)–(II) but which satisfies our requirement of having (2) as a boundary (sound) interpretation of "slightly true" in linearly ordered MV-algebras.

* * *

Introduce logic Ł_{st} as an extension of propositional Łukasiewicz logic (Ł) such that the language of Ł_{st} contains an additional unary connective "*st*" (the notion of a formula is extended to include formulas of the form $st\varphi$); Ł_{st} contains all axioms and deduction rules of Ł plus axioms

$$\varphi \to st\varphi, \tag{27}$$

$$(st\varphi \mathbin{\&} \neg st\psi) \to st\neg(\varphi \to \psi), \tag{28}$$

$$(st\neg\varphi \wedge st\neg\psi) \to st\neg(\varphi \vee \psi), \tag{29}$$

and the following deduction rule:

$$\frac{\neg\varphi}{\neg st\varphi}. \tag{30}$$

Described verbally, (28) says "if φ is slightly true and ψ is not (even) slightly true, then the implication $\varphi \to \psi$ is slightly false"; (29) says "if φ and ψ are slightly false, then their disjunction $\varphi \vee \psi$ is slightly false". Deduction rule (30) says: "from φ *is false* infer φ *is not even slightly true*". Axioms (27)–(29) and rule (30) seem to describe "natural properties" of "slightly true" in context of Łukasiewicz logic, however, this can be of course a matter of taste.

An *MV$_{st}$-algebra* $\mathbf{L}$ is an MV-algebra $\langle L, \cup, \cap, *, \Rightarrow, 0, 1\rangle$ (considered in the signature of residuated lattices) equipped with a unary operation $s\colon L \to L$ satisfying, for each $a, b \in L$,

$$s(0) = 0, \tag{31}$$

$$a \leq s(a), \tag{32}$$

$$s(a) * (s(b) \Rightarrow 0) \leq s((a \Rightarrow b) \Rightarrow 0), \tag{33}$$

$$s(a \Rightarrow 0) \cap s(b \Rightarrow 0) \leq s((a \cup b) \Rightarrow 0). \tag{34}$$

Remark 10. Each linearly ordered MV-algebra $\mathbf{L}$ with s given by (2) is an MV_{st}-algebra: (31) and (32) are obviously satisfied, (34) is satisfied in every chain, and (33) can be checked as follows. We have $s((a \Rightarrow b) \Rightarrow 0) = 0$ iff $(a \Rightarrow b) \Rightarrow 0 = 0$, which is true iff $a \Rightarrow b = 1$ iff $a \leq b$. Moreover, divisibility yields $s(b) * (s(b) \Rightarrow 0) = s(b) \cap 0 = 0$, i.e. $a \leq b$ gives $s(a) * (s(b) \Rightarrow 0) = 0$ because $*$ and s given by (2) are monotone. Therefore, $s((a \Rightarrow b) \Rightarrow 0) = 0$ implies $s(a) * (s(b) \Rightarrow 0) = 0$, showing (33). Hence, $\mathbf{L}$ equipped with (2) is indeed an MV_{st}-algebra. Note also that one can prove that s satisfying (31)–(34) is monotone (this will be shown in Remark 12). Altogether, s is a monotone superdiagonal function sending zero to zero—a desirable interpretation of “slightly true”.

We are now going to prove strong completeness of $Ł_{st}$ with respect to the semantics given by (linearly ordered) MV_{st}-algebras. The idea of the subsequent procedure is that we can define “particular fixed vt” inside $Ł_{st}$ and then we can use results on completeness of BL_{vt} [11] to prove completeness of $Ł_{st}$ itself.

We use the following notation. Throughout the rest of the paper, $Ł_{vt}$ denotes the extension of BL_{vt} which contains the additional axiom $\neg\neg\varphi \to \varphi$ (i.e., $Ł_{vt}$ is Łukasiewicz BL_{vt}-logic). For a formula φ of $Ł_{vt}$ define a formula $\lfloor\varphi\rfloor$ of $Ł_{st}$ as follows,

$$\lfloor\varphi\rfloor = \begin{cases} \overline{0} & \text{if } \varphi \text{ is } \overline{0}, \\ p & \text{if } \varphi \text{ is a propositional variable } p, \\ (\lfloor\chi\rfloor \to \lfloor\vartheta\rfloor) & \text{if } \varphi \text{ is } (\chi \to \vartheta), \\ (\lfloor\chi\rfloor \mathbin{\&} \lfloor\vartheta\rfloor) & \text{if } \varphi \text{ is } (\chi \mathbin{\&} \vartheta), \\ \neg st\neg\lfloor\psi\rfloor & \text{if } \varphi \text{ is } vt\psi. \end{cases}$$

Roughly speaking, $\lfloor\varphi\rfloor$ results from φ by replacing all its subformulas of the form $vt\psi$ by $\neg st\neg\psi$. Hence, $\lfloor\varphi\rfloor$ is indeed a formula of $Ł_{st}$. Dually, we define for each formula φ of $Ł_{st}$ a formula $\lceil\varphi\rceil$ of $Ł_{vt}$ which results by replacing subformulas $st\psi$ by $\neg vt\neg\psi$. Moreover, we extend the definitions of $\lfloor\cdots\rfloor$ and $\lceil\cdots\rceil$ on sets of formulas as follows: $\lceil T\rceil = \{\lceil\varphi\rceil \mid \varphi \in T\}$, $\lfloor T\rfloor = \{\lfloor\varphi\rfloor \mid \varphi \in T\}$. The following assertions present properties of formulas and theories flipped by $\lfloor\cdots\rfloor$ and $\lceil\cdots\rceil$:

Lemma 7. *$Ł_{vt}$ proves the following formulas,*

(i) $(\neg vt\neg\varphi \mathbin{\&} \neg\neg vt\neg\psi) \to \neg vt\neg\neg(\varphi \to \psi)$,
(ii) $(\neg vt\neg\neg\varphi \wedge \neg vt\neg\neg\psi) \to \neg vt\neg\neg(\varphi \vee \psi)$.
(iii) *Let T be theory over $Ł_{st}$, φ be a formula of $Ł_{st}$. If T proves φ over $Ł_{st}$, then $\lceil T\rceil$ proves $\lceil\varphi\rceil$ over $Ł_{vt}$.*

Proof.
“(i)”:
$\vdash vt(\varphi \to \psi) \to (\neg vt\neg\varphi \to \neg vt\neg\psi)$ [axiom (4)]
$\vdash \neg(\neg vt\neg\varphi \to \neg vt\neg\psi) \to \neg vt(\varphi \to \psi)$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$, MP]
$\vdash \neg(\neg vt\neg\varphi \to \neg vt\neg\psi) \to \neg vt\neg\neg(\varphi \to \psi)$ [using $\neg\neg\chi \equiv \chi$]

$\vdash \neg(\neg\neg vt\neg\psi \to \neg\neg vt\neg\varphi) \to \neg vt\neg\neg(\varphi \to \psi)$ [using $(\neg\chi \to \neg\vartheta) \equiv (\vartheta \to \chi)$]
$\vdash (\neg vt\neg\varphi \,\&\, \neg\neg vt\neg\psi) \to \neg vt\neg\neg(\varphi \to \psi)$ [using $\neg(\chi \to \neg\vartheta) \equiv (\vartheta \,\&\, \chi)$]

"(ii)":
$\vdash vt(\varphi \vee \psi) \to (vt\varphi \vee vt\psi)$ [axiom (5)]
$\vdash \neg(vt\varphi \vee vt\psi) \to \neg vt(\varphi \vee \psi)$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$, MP]
$\vdash (\neg vt\varphi \wedge \neg vt\psi) \to \neg vt(\varphi \vee \psi)$ [using $\neg(\chi \vee \vartheta) \equiv (\neg\chi \wedge \neg\vartheta)$]
$\vdash (\neg vt\neg\neg\varphi \wedge \neg vt\neg\neg\psi) \to \neg vt\neg\neg(\varphi \vee \psi)$ [using $\neg\neg\chi \equiv \chi$]

"(iii)": Observe that if γ is an axiom of $Ł_{st}$, then $\lceil\gamma\rceil$ is provable in $Ł_{vt}$. Indeed, if γ is (27), the claim is true due to Lemma 3 (i); if γ is (28) or (29), the claim follows from Lemma 7 (i) and (ii), respectively. If γ is an axiom of Ł then so is $\lceil\gamma\rceil$. Obviously, if T proves $\lceil\varphi\rceil$ and $\lceil\varphi \to \psi\rceil = \lceil\varphi\rceil \to \lceil\psi\rceil$ over $Ł_{vt}$, then T proves $\lceil\psi\rceil$ over $Ł_{vt}$. Moreover, if T proves $\lceil\neg\varphi\rceil$ over $Ł_{vt}$, then T proves $\lceil\neg st\varphi\rceil$ over $Ł_{vt}$: $T \vdash \lceil\neg\varphi\rceil = \neg\lceil\varphi\rceil$ yields $T \vdash vt\neg\lceil\varphi\rceil$ by (6), i.e., $T \vdash \neg\neg vt\neg\lceil\varphi\rceil = \neg\lceil st\varphi\rceil = \lceil\neg st\varphi\rceil$. The proof is finished by induction. □

Lemma 8. *$Ł_{st}$ proves the following formulas,*

(i) $\neg st\neg\varphi \to \varphi$,
(ii) $\neg st\neg(\varphi \to \psi) \to (\neg st\neg\varphi \to \neg st\neg\psi)$,
(iii) $\neg st\neg(\varphi \vee \psi) \to (\neg st\neg\varphi \vee \neg st\neg\psi)$.
(iv) *Let T be theory over $Ł_{vt}$, φ be a formula of $Ł_{vt}$. If T proves φ over $Ł_{vt}$, then $\lfloor T\rfloor$ proves $\lfloor\varphi\rfloor$ over $Ł_{st}$.*

Proof.
"(i)":
$\vdash \neg\varphi \to st\neg\varphi$ [axiom (27)]
$\vdash \neg st\neg\varphi \to \neg\neg\varphi$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$ and MP]
$\vdash \neg st\neg\varphi \to \varphi$ [by $\neg\neg\chi \to \chi$ and transitivity of implication]

"(ii)":
$\vdash (st\neg\psi \,\&\, \neg st\neg\varphi) \to st\neg(\neg\psi \to \neg\varphi)$ [axiom (28)]
$\vdash \neg st\neg(\neg\psi \to \neg\varphi) \to \neg(st\neg\psi \,\&\, \neg st\neg\varphi)$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$ and MP]
$\vdash \neg st\neg(\neg\psi \to \neg\varphi) \to (st\neg\psi \to st\neg\varphi)$ [using $\neg(\chi \,\&\, \neg\vartheta) \equiv (\chi \to \vartheta)$]
$\vdash \neg st\neg(\varphi \to \psi) \to (\neg st\neg\varphi \to \neg st\neg\psi)$ [using $(\chi \to \vartheta) \equiv (\neg\vartheta \to \neg\chi)$]

"(iii)":
$\vdash (st\neg\varphi \wedge st\neg\psi) \to st\neg(\varphi \vee \psi)$ [axiom (29)]
$\vdash \neg st\neg(\varphi \vee \psi) \to \neg(st\neg\varphi \wedge st\neg\psi)$ [by $(\chi \to \vartheta) \to (\neg\vartheta \to \neg\chi)$ and MP]
$\vdash \neg st\neg(\varphi \vee \psi) \to (\neg st\neg\varphi \vee \neg st\neg\psi)$ [using $\neg(\chi \wedge \vartheta) \equiv (\neg\chi \vee \neg\vartheta)$]

"(iv)": If γ is an axiom of $Ł_{vt}$, then $\lfloor\gamma\rfloor$ is an axiom of $Ł_{st}$, see Lemma 8 (i)–(iii). If T proves $\lfloor\varphi\rfloor$ over $Ł_{st}$, then T proves $\lfloor vt\varphi\rfloor$ over $Ł_{st}$. Indeed, if $T \vdash \lfloor\varphi\rfloor$ then $T \vdash \neg\neg\lfloor\varphi\rfloor$ from which we get $T \vdash \neg st\neg\lfloor\varphi\rfloor$ using (30), i.e. we have $T \vdash \neg st\neg\lfloor\varphi\rfloor = \lfloor vt\varphi\rfloor$. The rest is obvious. □

The following assertions, which are consequences of previous lemmas, put in correspondence the notions of provability in $Ł_{vt}$ and $Ł_{st}$.

Theorem 9. *Let T be theory over $Ł_{st}$, φ be a formula of $Ł_{st}$. Then T proves φ over $Ł_{st}$ iff $\lceil T\rceil$ proves $\lceil\varphi\rceil$ over $Ł_{vt}$.*

Proof. First, observe that due to the law of double negation, $Ł_{st}$ proves $\varphi \equiv \lfloor\lceil\varphi\rceil\rfloor$. As a consequence, if T is a theory over $Ł_{st}$, then T proves φ (over $Ł_{st}$) iff $\lfloor\lceil T\rceil\rfloor$ proves φ (over $Ł_{st}$). Therefore, we have that

$$T \text{ proves } \varphi \text{ (over } Ł_{st}) \quad \text{iff} \quad \lfloor\lceil T\rceil\rfloor \text{ proves } \lfloor\lceil\varphi\rceil\rfloor \text{ (over } Ł_{st}). \tag{35}$$

Now, the "$\Rightarrow$"-part of Theorem 9 follows from Lemma 7 (iii). Conversely, if $\lceil T \rceil$ proves $\lceil \varphi \rceil$ over Ł$_{vt}$, then, by Lemma 8 (iv), we have that $\lfloor \lceil T \rceil \rfloor$ proves $\lfloor \lceil \varphi \rceil \rfloor$ over Ł$_{st}$ which is true, due to (35), iff T proves φ over Ł$_{st}$, showing the "$\Leftarrow$"-part of Theorem 9. □

Corollary 10. *Let T be theory over Ł$_{vt}$, φ be a formula of Ł$_{vt}$. Then T proves φ over Ł$_{vt}$ iff $\lfloor T \rfloor$ proves $\lfloor \varphi \rfloor$ over Ł$_{st}$.*

Proof. T proves φ over Ł$_{vt}$ iff $\lceil \lfloor T \rfloor \rceil$ proves $\lceil \lfloor \varphi \rfloor \rceil$ over Ł$_{vt}$, which is true due to Theorem 9 iff $\lfloor T \rfloor$ proves $\lfloor \varphi \rfloor$ over Ł$_{st}$. □

We now turn our attention to MV$_{vt}$-algebras and MV$_{st}$-algebras. Recall that an MV$_{vt}$-algebra is a BL$_{vt}$-algebra satisfying $a = (a \Rightarrow 0) \Rightarrow 0$, i.e., MV$_{vt}$-algebra is an MV-algebra plus v satisfying (7)–(10).

For each MV$_{vt}$ algebra $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, v, 0, 1 \rangle$ we consider an algebra $\lfloor \mathbf{L} \rfloor = \langle L, \cup, \cap, *, \Rightarrow, s, 0, 1 \rangle$ where s is defined by $s(a) = v(a \Rightarrow 0) \Rightarrow 0$. Furthermore, for each $\mathbf{L}$-valuation e we consider an $\lfloor \mathbf{L} \rfloor$-evaluation $\lfloor e \rfloor$ which is uniquely given by $\lfloor e \rfloor(p) = e(p)$ (for each propositional variable p). Dually, for each MV$_{st}$ algebra $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, s, 0, 1 \rangle$ we consider an algebra $\lceil \mathbf{L} \rceil = \langle L, \cup, \cap, *, \Rightarrow, v, 0, 1 \rangle$ where v is defined by $v(a) = s(a \Rightarrow 0) \Rightarrow 0$; for each $\mathbf{L}$-valuation e we consider an $\lceil \mathbf{L} \rceil$-evaluation $\lceil e \rceil$ given by $\lfloor e \rfloor(p) = e(p)$. We now have:

Lemma 11. *Operators $\lfloor \cdots \rfloor$ and $\lceil \cdots \rceil$ satisfy the following properties:*

(i) *If $\mathbf{L}$ is an MV$_{vt}$-algebra and e is an $\mathbf{L}$-evaluation, then $\lfloor \mathbf{L} \rfloor$ is an MV$_{st}$-algebra and $\lfloor e \rfloor$ is an $\lfloor \mathbf{L} \rfloor$-evaluation such that, for each φ, $\lfloor e \rfloor(\lfloor \varphi \rfloor) = e(\varphi)$.*
(ii) *If $\mathbf{L}$ is an MV$_{st}$-algebra and e is an $\mathbf{L}$-evaluation, then $\lceil \mathbf{L} \rceil$ is an MV$_{vt}$-algebra and $\lceil e \rceil$ is an $\lceil \mathbf{L} \rceil$-evaluation such that, for each φ, $\lceil e \rceil(\lceil \varphi \rceil) = e(\varphi)$.*

Proof. "(i)": Suppose $\mathbf{L}$ is an MV$_{vt}$-algebra. By Lemma 4, $\mathbf{L}$ endowed by s defined by $s(a) = v(a \Rightarrow 0) \Rightarrow 0$ is an MV$_{vt,st}$-algebra, which yields that such an s satisfies (31) and (32). Since Ł$_{vt}$ is sound for all MV$_{vt}$-algebras, Lemma 7 gives $(s(a) * s(b \Rightarrow 0)) = ((v(a \Rightarrow 0) \Rightarrow 0) * ((v(b \Rightarrow 0) \Rightarrow 0) \Rightarrow 0)) \leq v(((a \Rightarrow b) \Rightarrow 0) \Rightarrow 0) \Rightarrow 0 = s((a \Rightarrow b) \Rightarrow 0)$, and $s(a \Rightarrow 0) \cap s(b \Rightarrow 0) = (v((a \Rightarrow 0) \Rightarrow 0) \Rightarrow 0) \cap (v((b \Rightarrow 0) \Rightarrow 0) \Rightarrow 0) \leq v(((a \cup b) \Rightarrow 0) \Rightarrow 0) \Rightarrow 0 = s((a \cup b) \Rightarrow 0)$, i.e. s satisfies (33) and (34). This proves that $\lfloor \mathbf{L} \rfloor$ is an MV$_{st}$-algebra.

It is easily seen that $\lfloor e \rfloor(\lfloor \overline{0} \rfloor) = \lfloor e \rfloor(\overline{0}) = 0 = e(\overline{0})$, and, for each propositional variable p, $\lfloor e \rfloor(\lfloor p \rfloor) = \lfloor e \rfloor(p) = e(p)$. Furthermore, $\lfloor e \rfloor(\lfloor \varphi \rightarrow \psi \rfloor) = \lfloor e \rfloor(\lfloor \varphi \rfloor \rightarrow \lfloor \psi \rfloor) = \lfloor e \rfloor(\lfloor \varphi \rfloor) \Rightarrow \lfloor e \rfloor(\lfloor \psi \rfloor) = e(\varphi) \Rightarrow e(\psi) = e(\varphi \rightarrow \psi)$, analogously for "&". Finally, we have $\lfloor e \rfloor(\lfloor vt\varphi \rfloor) = \lfloor e \rfloor(\neg st \neg \lfloor \varphi \rfloor) = s(\lfloor e \rfloor(\lfloor \varphi \rfloor) \Rightarrow 0) \Rightarrow 0 = s(e(\varphi) \Rightarrow 0) \Rightarrow 0 = v(e(\varphi)) = e(vt\varphi)$. Altogether, $\lfloor e \rfloor(\lfloor \varphi \rfloor) = e(\varphi)$ for each formula φ of Ł$_{vt}$.

"(ii)": Observe that Ł$_{st}$ is sound for MV$_{st}$-algebras, i.e. if $\mathbf{L}$ is a MV$_{st}$-algebra and if T proves φ over Ł$_{st}$, then $e(\varphi) = 1$ for each $\mathbf{L}$-model e of T. This is almost immediate: check axioms (27)–(29) and the corresponding inequalities (32)–(34); furthermore, if $e(\neg\varphi) = 1$, then $e(\varphi) = 0$, i.e., by (31), $s(e(\varphi)) = 0$, which gives $e(st\varphi) = 0$, thus $e(\neg st\varphi) = 1$. Now, soundness of Ł$_{st}$ and Lemma 8 (i)–(iii) yield that v defined by $v(a) = s(a \Rightarrow 0) \Rightarrow 0$ satisfies (8)–(10); (7) is satisfied because $v(1) = s(1 \Rightarrow 0) \Rightarrow 0 = s(0) \Rightarrow 0 = 0 \Rightarrow 0 = 1$. Therefore, $\lceil \mathbf{L} \rceil$ is a MV$_{vt}$-algebra. Claim $\lceil e \rceil(\lceil \varphi \rceil) = e(\varphi)$ can be proved analogously as in case of (i). □

Theorem 12. *Let T be a theory over Ł$_{st}$, φ a formula of Ł$_{st}$. The following statements are equivalent.*

(i) *For each* (*linear*) *MV_{st}-algebra* $\mathbf{L}$ *and each* $\mathbf{L}$*-model* e *of* T, $e(\varphi) = 1$.
(ii) *For each* (*linear*) *MV_{vt}-algebra* $\mathbf{L}$ *and each* $\mathbf{L}$*-model* e *of* $\lceil T \rceil$, $e(\lceil \varphi \rceil) = 1$.

Proof. First, if $\mathbf{L}$ is a linear MV_{st}-algebra (MV_{vt}-algebra), then $\lceil \mathbf{L} \rceil$ ($\lfloor \mathbf{L} \rfloor$) is a linear MV_{vt}-algebra (MV_{st}-algebra). Now,

"$\Rightarrow$": Assume (i) is true and let e be an $\mathbf{L}$-model of $\lceil T \rceil$. Using Lemma 11 (i), $\lfloor e \rfloor$ is an $\lfloor \mathbf{L} \rfloor$-model of $\lfloor \lceil T \rceil \rfloor$. Since, for each formula ψ of $Ł_{st}$, $\lfloor e \rfloor(\psi) = \lfloor e \rfloor(\lfloor \lceil \psi \rceil \rfloor)$, we get that $\lfloor e \rfloor$ is an $\lfloor \mathbf{L} \rfloor$-model of T. Applying (i), we get $\lfloor e \rfloor(\varphi) = 1$, which further gives $\lfloor e \rfloor(\lfloor \lceil \varphi \rceil \rfloor) = 1$. Using Lemma 11 (i), we get $e(\lceil \varphi \rceil) = 1$, showing (ii).

"$\Leftarrow$": Suppose (ii) holds and let e be an $\mathbf{L}$-model of T. Lemma 11 (ii) gives that $\lceil e \rceil$ is an $\lceil \mathbf{L} \rceil$-model of $\lceil T \rceil$. By (ii) and Lemma 11 (ii), $e(\varphi) = \lceil e \rceil(\lceil \varphi \rceil) = 1$. □

Theorem 13 (strong completeness of $Ł_{st}$)**.** *Let T be a theory over $Ł_{st}$, φ a formula. The following are equivalent:*

(i) *T proves φ over $Ł_{st}$.*
(ii) *For each* (*linearly ordered*) *MV_{st}-algebra* $\mathbf{L}$ *and each* $\mathbf{L}$*-model* e *of* T, $e(\varphi) = 1$.

Proof. By Theorem 9, T proves φ over $Ł_{st}$ iff $\lceil T \rceil$ proves $\lceil \varphi \rceil$ over $Ł_{vt}$, which is true, due to completeness of $Ł_{vt}$, iff for each (linearly ordered) MV_{vt}-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of $\lceil T \rceil$, $e(\lceil \varphi \rceil) = 1$. Applying Theorem 12, the latter claim is true iff for each (linearly ordered) MV_{st}-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T, $e(\varphi) = 1$. □

Remark 11. In Section 3, we introduced $BL_{vt,st}$-algebras as extensions of BL_{vt}-algebras: each BL_{vt}-algebra can be extended to an $BL_{vt,st}$-algebra, and the reduct of a $BL_{vt,st}$-algebra which results by removing the truth-depressing hedge is a BL_{vt}-algebra. The reduct of an $MV_{vt,st}$-algebra which results by removing the *truth-stressing* hedge need not be an MV_{st}-algebra in general (for instance, consider $\mathbf{L}$ from Example 8 with v_1 and s_2). On the other hand, the following corollary shows that each MV_{st}-algebra can be extended to an $MV_{vt,st}$-algebra:

Corollary 14. *Let* $\mathbf{L} = \langle L, \cup, \cap, *, \Rightarrow, s, 0, 1 \rangle$ *be an MV_{st}-algebra,* $v : L \to L$ *be defined by* $v(a) = s(a \Rightarrow 0) \Rightarrow 0$. *Then*

(i) $\mathbf{L}$ *equipped with* v *is an $MV_{vt,st}$-algebra;*
(ii) *if* $\mathbf{L}$ *equipped with* $v' : L \to L$ *is an $MV_{vt,st}$-algebra, then* $v'(a) \leq v(a)$ $(a \in L)$.

Proof. "(i)": By Lemma 11 (ii), $\lceil \mathbf{L} \rceil$ is an MV_{vt}-algebra where v is given by $v(a) = s(a \Rightarrow 0) \Rightarrow 0$. Thus, v is a truth-stressing hedge on MV-algebra $\langle L, \cup, \cap, *, \Rightarrow, 0, 1 \rangle$. From $v(a) = s(a \Rightarrow 0) \Rightarrow 0$ we get $v(a) \Rightarrow 0 = s(a \Rightarrow 0)$ and $v(a \Rightarrow 0) \Rightarrow 0 = s(a)$. Hence, Lemma 4 (iv) gives that s is a truth-depressing hedge associated with v. Altogether, $\mathbf{L}$ with v is an $MV_{vt,st}$-algebra.

"(ii)": Let $\mathbf{L}$ with v' be an $MV_{vt,st}$-algebra. Since $st\neg\varphi \to \neg vt\neg\neg\varphi$ is an axiom of $Ł_{vt,st}$, $s(a \Rightarrow 0) \leq v'((a \Rightarrow 0) \Rightarrow 0) \Rightarrow 0$ because $Ł_{vt,st}$ is sound for $\mathbf{L}$ with v'. Therefore, $s(a \Rightarrow 0) \leq v'(a) \Rightarrow 0$, by adjointness, $v'(a) \leq s(a \Rightarrow 0) \Rightarrow 0 = v(a)$, which is the desired inequality. □

Remark 12. Let us mention that Corollary 14 (i) yields that each unary function s satisfying (31)–(34) is monotone, cf. Remark 10. Moreover, Corollary 14 (ii) says that truth-stressing hedge v given by $v(a) = s(a \Rightarrow 0) \Rightarrow 0$ is among all the truth-stressing hedges v' with property "s is associated with v'", the greatest one. This observation is interesting because as we have seen in Lemma 4, s defined by $s(a) = v(a \Rightarrow 0) \Rightarrow 0$ is the greatest truth-depressing hedge associated with v.

6. Applications in fuzzy relational systems

Hedges can be used to control interpretation of IF-THEN rules, this idea was used, e.g., in [4, 6, 7]. Motivation is the following. We consider particular compound formulas of the form $\varphi \to \psi$, where φ and ψ are formulas (of some language) and $\to$ is a symbol for logical connective "fuzzy implication". Thus, we deal with pairs of formulas connected in an implicative manner. Our primary interpretation of $\varphi \to \psi$ is "if φ then ψ". In practical applications, it may be desirable to fine-tune the interpretation with additional (linguistic) hedges: $\varphi \to \psi$ is interpreted as "if (very/slightly/$\cdots$) φ then (very/slightly/$\cdots$) ψ". In [4, 6, 7], we used interpretation "if very φ then ψ". Employing truth-depressing hedges, we get other modifications of the interpretation of IF-THEN rules. The interpretation "if very φ then ψ" turned out to be suitable for problems studied in [4, 6, 7]. For instance, in [4] we dealt with data tables with fuzzy attributes (particular object-attribute data sets describing degrees to which "attributes apply to objects") and their non-redundant bases. A non-redundant basis of a given table is, roughly speaking, a minimal set of IF-THEN rules (interpreted "if very φ then ψ") which fully describe all dependencies which are true the data table. In [4, 6] we showed that the choice of the interpretation of "very" may affect, e.g., existence and uniqueness of non-redundant bases which can be efficiently computed. The detailed description of the topic is outside the scope of this paper, see [4, 6] for details.

Another motivation for using hedges as parameters of the interpretation of formulas is, in fact, closely connected with the classical interpretation of (ordinary) IF-THEN rules. If we denote truth degrees of φ and ψ (in some interpretation) by $||\varphi||$ and $||\psi||$, respectively, we usually define a truth-degree of the compound formula $||\varphi \to \psi||$. In classical (two-valued) setting, we define $||\varphi \to \psi||$ using one of the following definitions which are all equivalent (in classical setting):

(i) Set $||\varphi \to \psi||$ to $||\varphi|| \Rightarrow ||\psi||$, where $\Rightarrow$ is a binary truth function defined by $(0 \Rightarrow 0) = (0 \Rightarrow 1) = (1 \Rightarrow 1) = 1$, and $(1 \Rightarrow 0) = 0$.
(ii) If $||\varphi|| = 1$, then set $||\varphi \to \psi||$ to $||\psi||$; if $||\varphi|| \neq 1$, set $||\varphi \to \psi||$ to 1.
(iii) If $||\psi|| = 1$, set $||\varphi \to \psi||$ to 1; if $||\psi|| = 0$, set $||\varphi \to \psi||$ to negated $||\varphi||$.
(iv) If $||\varphi|| = 1$ and $||\psi|| = 0$, set $||\varphi \to \psi||$ to 0; otherwise set $||\varphi \to \psi||$ to 1.

Now, if we want to introduce $||\varphi \to \psi||$ in fuzzy setting, a straightforward (and most commonly) used way is to put $||\varphi \to \psi|| = ||\varphi|| \Rightarrow ||\psi||$, where $\Rightarrow$ is (usually) a (truth function of) residuated implication. This corresponds to (i) from the previous list, we only have replaced truth function of the classical implication by a general residuated one. A question is, if (ii)–(iv) have also (reasonable) translations and generalizations in fuzzy setting. One way to proceed is via hedges. Suppose $\mathbf{L}$ is a linearly ordered $\mathrm{BL}_{vt,st}$-algebra with v and s given by (1) and (2), respectively. Then, for any formulas φ, ψ of $\mathrm{BL}_{vt,st}$ and their interpretations $||\varphi||$ and $||\psi||$ (i.e., $||\chi|| = e(\chi)$ for certain $\mathbf{L}$-model e), we have

$$||vt\varphi \to \psi|| = \begin{cases} ||\psi|| & \text{if } ||\varphi|| = 1, \\ 1 & \text{otherwise,} \end{cases} \tag{36}$$

$$||\varphi \to st\psi|| = \begin{cases} ||\varphi|| \Rightarrow 0 & \text{if } ||\psi|| = 0, \\ 1 & \text{otherwise,} \end{cases} \tag{37}$$

$$||vt\varphi \to st\psi|| = \begin{cases} 0 & \text{if } ||\varphi|| = 1 \text{ and } ||\psi|| = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{38}$$

As one can see, (36)–(38) generalize the above-mentioned classical interpretations (ii)–(iv). By various choices of v and s, i.e. other than (1) and (2), we obtain further generalizations of (ii)–(iv) in fuzzy setting which can be read: "if very φ then ψ", "if φ then slightly ψ", and "if very φ then slightly ψ". Note also that $||vt\varphi \to st\psi||$ is from that point of view the most general one because (i)–(iv) result by borderline choices of v and s. There are, of course, other possible placements of "st" and "vt" in IF-THEN rules, e.g. $vt\varphi \to vt\psi$, $st\varphi \to vt\psi, \ldots$ Our placements defined by (36)–(38) stress the truth of $\varphi \to \psi$: due to antitony of residuum in its first argument, monotony in the second one, and due to subdiagonality (superdiagonality) of v (s), we always have $||\varphi \to \psi|| \leq ||vt\varphi \to st\psi||$.

* * *

Truth-stressing hedges turned out to be useful in fuzzy concept analysis. Idempotent truth-stressing hedges can be used to control, in a parameterized way, the size of a fuzzy concept lattice (a complete lattice structure of clusters extracted from data with fuzzy attributes), see [3, 5]. Form the theoretical point of view, hedges are employed as parameters of Galois connections and closure operators. In the rest of this section we show that truth-depressing hedges as defined in Section 3 are closely related to certain fuzzy closure operators which are of interest in fuzzy concept analysis.

In order to be consistent with [3, 5], we slightly change our notation. Our basic structures of truth degrees will be complete residuated lattices [2] and will be denoted by $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$. Caution: unlike previous chapters, $\to$, $\wedge$, and $\vee$ stand for operations of $\mathbf{L}$ (residuum, meet, and join) and not for symbols of logical connectives—this is for consistency with [3, 5] (we do not use symbols of logical connectives anymore). Given $\mathbf{L}$, an $\mathbf{L}$-set (fuzzy sets with truth degrees in $\mathbf{L}$) A in universe X is a mapping $A\colon X \to L$, $A(x)$ being interpreted as "degree to which x belongs to A". By $\{^{a}/x\}$ we denote an $\mathbf{L}$-set in X such that $\{^{a}/x\}(x) = a$ and $\{^{a}/x\}(y) = 0$ ($y \neq x$). The empty $\mathbf{L}$-set in X is denoted by $\emptyset$, i.e. $\emptyset(x) = 0$ ($x \in X$). The collection of all $\mathbf{L}$-sets in X is denoted by L^X. For $A, B \in L^X$, we write $A \subseteq B$ iff, for each $x \in X$, $A(x) \leq B(x)$, see [2].

Let $\mathbf{L}$ be a linearly ordered complete residuated lattice, v be an idempotent truth-stressing hedge on $\mathbf{L}$, i.e. let v satisfy (7)–(9) plus $v(v(a)) = v(a)$ ($a \in L$). An operator $\mathrm{C}\colon L^X \to L^X$ (i.e., an operator on fuzzy sets in universe X) is called a *closure operator with truth-stressing hedge* v [5] if, for each $A, B \in L^X$,

$$A \subseteq \mathrm{C}(A), \tag{39}$$

$$v(S(A,B)) \leq S(\mathrm{C}(A), \mathrm{C}(B)), \tag{40}$$

$$\mathrm{C}(\mathrm{C}(A)) \subseteq \mathrm{C}(A), \tag{41}$$

where $S(A,B) \in L$ denotes *degree of subsethood of* A *in* B, which is defined by $S(A,B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Observe that in (40), the hedge v is used as a parameter of the monotony condition, which can be read as: "if it is very true that A is a subset of B, then the closure of A is a subset of the closure of B", see [5]. Fuzzy closure operators with hedges play an important role in fuzzy concept analysis and fuzzy attribute logic, for more details see [3, 4, 5].

The following assertion shows a correspondence between certain fuzzy closure operators with truth-stressing hedge v (namely, operators which are defined coordinatewise by unary functions) and truth-depressing hedges associated with v.

Observation 15. *Let* $\{s_x \colon L \to L \mid x \in X\}$ *be a system of unary functions and let* $\mathrm{C}\colon L^X \to L^X$ *be an operator given by* $(\mathrm{C}(A))(x) = s_x(A(x))$ $(x \in X,\ A \in L^X)$. *Suppose* $\mathrm{C}(\emptyset) = \emptyset$. *Then* C *is a fuzzy closure operator with truth-stressing hedge* v *iff each* s_x *is an idempotent truth-depressing hedge associated with* v.

Proof. "$\Rightarrow$": Let C be a fuzzy closure operator with hedge v. We show that each s_x is an idempotent truth-depressing hedge associated with v. Since $\mathrm{C}(\emptyset) = \emptyset$, we get $s_x(0) = s_x(\emptyset(x)) = (\mathrm{C}(\emptyset))(x) = \emptyset(x) = 0$, i.e. each s_x satisfies (16). Furthermore, (39) yields $a = \{{}^a/x\}(x) \leq (\mathrm{C}(\{{}^a/x\}))(x) = s_x(\{{}^a/x\}(x)) = s_x(a)$, showing (17). Using (40),

$$\begin{aligned} v(a \to b) &= v(\{{}^a/x\}(x) \to \{{}^b/x\}(x)) = v(S(\{{}^a/x\}, \{{}^b/x\})) \leq \\ &\leq S(\mathrm{C}(\{{}^a/x\}), \mathrm{C}(\{{}^b/x\})) \leq (\mathrm{C}(\{{}^a/x\}))(x) \to (\mathrm{C}(\{{}^b/x\}))(x) = \\ &= s_x(\{{}^a/x\}(x)) \to s_x(\{{}^b/x\}(x)) = s_x(a) \to s_x(b), \end{aligned}$$

i.e. each s_x is a truth-depressing hedge associated with v. Moreover, s_x is idempotent: using (41), we obtain $s_x(s_x(a)) = s_x((\mathrm{C}(\{{}^a/x\}))(x)) = (\mathrm{C}(\mathrm{C}(\{{}^a/x\})))(x) \leq (\mathrm{C}(\{{}^a/x\}))(x) = s_x(\{{}^a/x\}(x)) = s_x(a)$.

"$\Leftarrow$": Let s_x $(x \in X)$ be idempotent truth-depressing hedges associated with v. We check that C is a closure operator with v. (39) is a direct consequence of (17): $A(x) \leq s_x(A(x)) = (\mathrm{C}(A))(x)$. (40) is a consequence of (18) because

$$\begin{aligned} v(S(A, B)) &= v(\textstyle\bigwedge_{x\in X}(A(x) \to B(x))) \leq \bigwedge_{x\in X} v(A(x) \to B(x)) \leq \\ &\leq \textstyle\bigwedge_{x\in X}(s_x(A(x)) \to s_x(B(x))) = \bigwedge_{x\in X}((\mathrm{C}(A))(x) \to (\mathrm{C}(B))(x)) = \\ &= S(\mathrm{C}(A), \mathrm{C}(B)). \end{aligned}$$

Finally, (41) holds due to idempotency of each s_x: $(\mathrm{C}(\mathrm{C}(A)))(x) = s_x(s_x(A(x))) = s_x(A(x)) = (\mathrm{C}(A))(x)$. □

7. Conclusions and open problems

The main aim of this paper was to show that (i) truth-depressing hedges are interesting from the pure logico-algebraic point of view, and that (ii) truth-depressing hedges considered as particular superdiagonal monotone truth functions may also be of interest in fuzzy logic in wider sense and in applications. We focused on two approaches. The first one introduced "*st*" relatively to "*vt*". The second one showed a possibility to axiomatize "*st*" in Gödel, Łukasiewicz, and product logics. The following list contains some open problems.

- Axiomatization of "*st*" in Section 3 relies on the axiomatization of "*vt*" as introduced in [11]; it may be interesting to introduce "*st*" in weaker logics than BL and/or look at "*st*" from the point of view of modalities in fuzzy logic, see also [8, 15].
- Find an axiomatization of "*st*" without introducing "*vt*" (or Baaz's $\triangle$) so that (i) truth function (2) would be a borderline interpretation of "*st*" in each linearly ordered structure of truth degrees and, at the same time, (ii) connective "*st*" would be interpreted by (some) nonidempotent truth functions (like $\sqrt{a}$ on $[0, 1]$).
- We did not discuss issues related with standard completeness. This may be of some interest, cf. also [11, 12].

8. Acknowledgment

Supported by grant no. B1137301 of the Grant Agency of the Academy of Sciences of Czech Republic, grant No. 201/05/0079 of the Czech Science Foundation, and by institutional support, research plan MSM 6198959214. My thanks go to both the anonymous referees for valuable comments and suggestions.

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