Reducing the size of fuzzy concept lattices by fuzzy closure operators

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***Abstract*— The paper presents a general method of imposing constraints in formal concept analysis of tabular data with fuzzy attributes. The constraints represent a user-defined requirements which are supplied along with the input data table. The main effect is to filter-out conceptual clusters (outputs of the analysis) which are not compatible with the constraint, in a computationally efficient way. Our approach covers several examples studied before, e.g. crisply generated concepts and constraints by hedges.**

***Keywords*— formal concept analysis, fuzzy attribute, tabular data, constraint, fuzzy closure operator, fuzzy concept lattice**

I. Introduction and Problem Setting

Formal concept analysis (FCA) is a method of data analysis and visualization which deals with input data in the form of a table describing objects (rows), their attributes (columns), and their relationship. In the basic setting, the relationship is a bivalent one. That is, attributes are bivalent meaning that a given object either has (indicated by 1 in the corresponding table entry) or does not have (indicated by 0) an attribute. FCA with bivalent attributes is well described, e.g., in [5], [6]. It is often the case that attributes are fuzzy (graded) rather than bivalent. That is, a given object has a given attribute to a given degree. For instance, a lecture may be considered well organized to degree 0.8. In order to be able to deal with fuzzy attributes, FCA has been accordingly extended, see e.g. [1], [11].

The basic output of FCA is a collection of so-called formal concepts. Formal concepts are interpreted as conceptual clusters and can be partially ordered under a natural conceptual hierarchy. The resulting partially ordered collection of clusters is called a concept lattice. FCA proved to be useful in several fields either as a direct method of data analysis or as a preprocessing method. It is assumed that no further information is supplied at the input except for the data table. However, it is often the case that there is an additional information available in the form of a constraint (requirement) specified by a user. In such a case, one is not interested in all the outputs but only in those which satisfy the constraint. The other outputs may be left out as non-interesting. This way, the number of output clusters is reduced by focusing on the "interesting ones".

In this paper, we develop a general method of constraints in formal concept analysis of data with fuzzy attributes. The constraints are expressed by means of closure operators. Our approach is theoretically and computationally tractable and covers several interesting forms of constraints. In Section II we survey preliminaries on fuzzy logic. Section III provides an overview of formal concept analysis of data with fuzzy attributes. Our approach and main results are contained in Section IV. Section V presents various examples of constraints. In Section VI, we present examples of constrained concept lattices. Section VII discusses selected further issues.

II. Preliminaries from Fuzzy Logic

In this section we present basic notions of fuzzy logic and fuzzy sets. More details can be found in monographs [1], [7]. A complete residuated lattice, which is our basic structure of truth degrees, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy so-called adjointness property [1], [7]. Each $a \in L$ is called a truth degree; $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Complete residuated lattices include structures of truth degrees defined on the real unit interval with $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Finite residuated lattices represent another important subfamily of complete residuated lattices. A particular finite residuated lattice is the Boolean algebra with $L = \{0, 1\}$ (structure of truth degrees of classical logic). Given $\mathbf{L}$ which serves as a structure of truth degrees, we introduce the usual structural notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". $\mathbf{L}^U$ denotes the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ ($u \in U$). Binary $\mathbf{L}$-relations (binary fuzzy relations) between U and V can be thought of as $\mathbf{L}$-sets in $U \times V$. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A, B) = \bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big)$ which generalizes the classical subsethood relation $\subseteq$ in a fuzzy setting. Subsethood degree $S(A, B) \in L$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ (A is fully included in B) if $S(A, B) = 1$. Under this notation, we have that $A \subseteq B$ iff, for each $u \in U$, $A(u) \leq B(u)$.

III. Formal Concept Analysis of Data with Fuzzy Attributes

This section summarizes basic notions of formal concept analysis of data with fuzzy attributes.

A *data table with fuzzy attributes*, which represents the input object-attribute data table, is represented by a triplet $\langle X, Y, I\rangle$ where X is a set of objects, Y is a finite set of attributes, and $I \in \mathbf{L}^{X \times Y}$ is a binary fuzzy relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x, y) \in L$ to which x has y. $\langle X, Y, I\rangle$ can be thought of as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$.

For $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^{\uparrow} \in \mathbf{L}^Y$ (fuzzy set of attributes), $B^{\downarrow} \in \mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\uparrow}(y) = \bigwedge_{x\in X}\big(A(x) \to I(x, y)\big),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y}\big(B(y) \to I(x, y)\big).$$

Described verbally, $A^{\uparrow}$ is the fuzzy set of all attributes from Y shared by all objects from A (and similarly for $B^{\downarrow}$). A *fuzzy concept* (*conceptual cluster*) in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$. That is, a fuzzy concept consists of a fuzzy set A (so-called *extent*) of objects which fall under the concept and a fuzzy set B (so-called *intent*) of attributes which fall under the concept such that A is the fuzzy set of all objects sharing all attributes from B and, conversely, B is the fuzzy set of all attributes from Y shared by all objects from A.

A collection $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ of all conceptual clusters in $\langle X, Y, I\rangle$ can be equipped with a partial order $\leq$ modeling the subconcept-superconcept hierarchy (e.g., *dog* $\leq$ *mammal*) defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \quad (1)$$

Note that $^{\uparrow}$ and $^{\downarrow}$ form a so-called fuzzy Galois connection [1] and that $\mathcal{B}(X, Y, I)$ is in fact a set of all fixed points of $^{\uparrow}$ and $^{\downarrow}$. Under $\leq$, $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called a *fuzzy concept lattice* of $\langle X, Y, I\rangle$. The basic structure of fuzzy concept lattices is described by the so-called main theorem of concept lattices [1] whose first part is the following.

Theorem 1 (see [1]): The set $\mathcal{B}(X, Y, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle,$$
$$\bigvee_{j\in J}\langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle. \quad \blacksquare$$

Recall that a *fuzzy closure operator* on a set U is a mapping $C : \mathbf{L}^U \to \mathbf{L}^U$ satisfying (i) $B \subseteq C(B)$ (extensivity); (ii) if $B_1 \subseteq B_2$ then $C(B_1) \subseteq C(B_2)$ (monotony); and (iii) $C(C(B)) = C(B)$ (idempotency). The composed operators $^{\uparrow\downarrow}$ and $^{\downarrow\uparrow}$ are fuzzy closure operators on X and Y, respectively. In the sequel we denote by $\mathrm{Int}(X, Y, I)$ the set of all intents of $\mathcal{B}(X, Y, I)$. Thus, $\mathrm{Int}(X, Y, I) \subseteq \mathbf{L}^Y$ such that $B \in \mathbf{L}^Y$ belongs to $\mathrm{Int}(X, Y, I)$ iff there is $A \in \mathbf{L}^X$ such that $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$. In other words, $\mathrm{Int}(X, Y, I)$ is the set of all fixed points of fuzzy closure operator $^{\downarrow\uparrow}$.

For a detailed information on formal concept analysis of data tables with fuzzy attributes we refer to [1], [2]. Formal concept analysis of data tables with binary attributes is thoroughly studied in [5], [6] where a reader can find theoretical foundations, methods and algorithms, and applications in various areas.

IV. Constraints by Fuzzy Closure Operators

Selecting "interesting" clusters from $\mathcal{B}(X, Y, I)$ needs to be accompanied by a criterion of what is interesting. Such a criterion can be seen as a constraint and depends on particular data and application. Therefore, the constraint should be supplied by a user along with the input data table with fuzzy attributes. One way to specify "interesting concepts" is to focus on concepts whose fuzzy sets of attributes are "interesting". This seems to be natural because "interesting concepts" are determined by "interesting attributes/properties of objects". Thus, for a data table with fuzzy attributes $\langle X, Y, I\rangle$, the user may specify a collection $Y' \subseteq \mathbf{L}^Y$ of fuzzy sets of attributes such that $B \in Y'$ iff the user considers B to be an interesting fuzzy set of attributes. A conceptual cluster $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ can be then seen as "interesting" if $B \in Y'$. In this section we develop this idea provided that the selected fuzzy sets of attributes which are taken as "interesting" can be seen as fixed points of a fuzzy closure operator on Y. Thus, we may introduce the concept of an interesting fuzzy set of attributes as follows.

Let Y be a set of attributes, $C : \mathbf{L}^Y \to \mathbf{L}^Y$ be a fuzzy closure operator on Y. A fuzzy set $B \in \mathbf{L}^Y$ of attributes is called a *C-interesting fuzzy set of attributes* if $B = C(B)$.

Remark 1: (1) Representing interesting fuzzy sets of attributes by closure operators has technical as well as epistemic reasons. Specifying particular C, we prescribe a particular meaning of "being interesting". Given a fuzzy set $B \in \mathbf{L}^Y$ of attributes, either we have $B = C(B)$, i.e. B is C-interesting, or $B \subset C(B)$ which can be read: "B is not C-interesting, but additional attributes would make B interesting". Hence, $C(B)$ should be understood as *the least fuzzy set of C-interesting attributes containing B*.

(2) A definition of C depends on a particular application. In our approach, we assume that C is any fuzzy closure operator, covering thus all possible choices of C. On the other hand, in practical applications, it is necessary to have a collection of easy-to-understand definitions of such fuzzy closure operators. In Section V we give several examples to define C which are intuitively clear for an inexperienced user.

Let $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, C be a fuzzy closure operator on Y. We put

$$\mathcal{B}_C(X, Y, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid B = C(B)\},$$
$$\mathrm{Int}_C(X, Y, I) = \{B \in \mathrm{Int}(X, Y, I) \mid B = C(B)\}.$$

Each $\langle A, B\rangle \in \mathcal{B}_C(X, Y, I)$ will be called a *C-interesting fuzzy concept* (in short, a *C-concept*); each $B \in \mathrm{Int}_C(X, Y, I)$ will be called a *C-interesting intent* (in short, a *C-intent*).

Remark 2: (1) By definition, $\langle A, B\rangle$ is a C-concept iff $\langle A, B\rangle$ is a fuzzy concept (i.e. $A^{\uparrow} = B$ and $B^{\downarrow} = A$) such that B is a C-interesting fuzzy set of attributes. Notice that two boundary cases of closure operators on Y are (i) $C(B) = B$ ($B \in \mathbf{L}^Y$), (ii) $C(B) = Y$ ($B \in \mathbf{L}^Y$), i.e., for each $B \in \mathbf{L}^Y$ and $y \in Y$, we have $(C(B))(y) = 1$. For C defined by (i), the notion of a C-concept coincides with that of a fuzzy concept. In this case, $\mathcal{B}_C(X, Y, I)$ equals $\mathcal{B}(X, Y, I)$. In case of (ii), $\mathcal{B}_C(X, Y, I)$ is a one-element set (not interesting).

(2) Observe that B is a C-intent iff $B = B^{\downarrow\uparrow} = C(B)$. Denoting the set of all fixed points of C by $\mathrm{fix}(C)$, we have $\mathrm{Int}_C(X, Y, I) = \mathrm{Int}(X, Y, I) \cap \mathrm{fix}(C)$.

The structure of C-concepts if characterized by

Theorem 2: Let $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, C be a fuzzy closure operator on Y. Then $\mathcal{B}_C(X, Y, I)$ equipped with $\leq$ defined by (1) is a complete lattice which is a $\bigvee$-sublattice of $\mathcal{B}(X, Y, I)$.

Proof: In order to show that $\mathcal{B}_C(X, Y, I)$ equipped with $\leq$ is a complete lattice, it suffices to check that Int_C is closed under arbitrary infima. Take a system $\{B_i \in \mathrm{Int}_C(X, Y, I) \mid i \in I\}$ of C-intents. Since $B_i \in \mathrm{Int}(X, Y, I)$, Theorem 1 yields $\bigcap_{i\in I} B_i \in \mathrm{Int}(X, Y, I)$. Now, it remains to show that $B = \bigcap_{i\in I} B_i$ is closed under C. Since each B_i is closed under C, we get $B = \bigcap_{i\in I} B_i = \bigcap_{i\in I} C(B_i) = C\big(\bigcap_{i\in I} C(B_i)\big) = C\big(\bigcap_{i\in I} B_i\big) = C(B)$. Altogether, $B \in \mathrm{Int}_C(X, Y, I)$. To see that $\mathcal{B}_C(X, Y, I)$ is a $\bigvee$-sublattice of $\mathcal{B}(X, Y, I)$ observe that $\mathrm{Int}(X, Y, I)$ and $\mathrm{Int}_C(X, Y, I)$ agree on arbitrary intersections and then apply Theorem 1. ■

We now focus on the computational aspects of generating all C-concepts. The naive way to compute $\mathcal{B}_C(X, Y, I)$ is to find $\mathcal{B}(X, Y, I)$ first and then go through all of its concepts and filter out the C-concepts. This method is not efficient because in general, $\mathcal{B}_C(X, Y, I)$ can be considerably smaller than $\mathcal{B}(X, Y, I)$. In the sequel we outline a way to compute $\mathcal{B}_C(X, Y, I)$ directly without the need to compute $\mathcal{B}(X, Y, I)$. For any $B \in \mathbf{L}^Y$ define fuzzy sets B_i ($i \in \mathbb{N}_0$) and $\mathcal{C}(B)$ of attributes as follows:

$$B_i = \begin{cases} B & \text{if } i = 0, \\ C({B_{i-1}}^{\downarrow\uparrow}) & \text{if } i \geq 1. \end{cases} \quad (2)$$

$$\mathcal{C}(B) = \textstyle\bigcup_{i=1}^{\infty} B_i. \quad (3)$$

Theorem 3: Let $\mathbf{L}$ be a finite residuated lattice, Y be a finite set of attributes, $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, C be a fuzzy closure operator on Y, $\mathcal{C}$ be defined by (3). Then $\mathcal{C}$ is a fuzzy closure operator such that $B = \mathcal{C}(B)$ iff $B \in \mathrm{Int}_C(X, Y, I)$.

Proof: Extensivity of $\mathcal{C}$ follows directly from extensivity of ${}^{\downarrow\uparrow}$ and C. It is also easily seen that $\mathcal{C}$ is monotone. To check idempotency of $\mathcal{C}$, for each $B \in \mathbf{L}^Y$, we show $C(((\mathcal{C}(B))^{\downarrow\uparrow}) \subseteq \mathcal{C}(B)$; the converse inclusion follows from extensivity of $\mathcal{C}$. Observe that $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$, i.e. the sequence defined by (2), is non-decreasing. Since both $\mathbf{L}$ and Y are finite, we get that $\mathbf{L}^Y$ is finite. Moreover, since each B_i defined by (2) is in $\mathbf{L}^Y$, we get that there must be an index $i \in \mathbb{N}_0$ such that $B_i = B_j$ ($j \leq i$). Thus, we have $\mathcal{C}(B) = B_i$, i.e. $C(((\mathcal{C}(B))^{\downarrow\uparrow}) = C({B_i}^{\downarrow\uparrow}) = B_{i+1} \subseteq \mathcal{C}(B)$, i.e. $\mathcal{C}$ is idempotent. Altogether, $\mathcal{C}$ is a fuzzy closure operator. We now prove that B is closed under $\mathcal{C}$ iff $B \in \mathrm{Int}_C(X, Y, I)$. "⇒": Let $B = \mathcal{C}(B)$. Using the above idea, $\mathcal{C}(B) = B_i$ for some $i \in I$. Therefore, $B = B_i = C({B_{i-1}}^{\downarrow\uparrow})$, showing $B = C(B)$. Moreover, $B^{\downarrow\uparrow} = {B_i}^{\downarrow\uparrow} \subseteq C({B_i}^{\downarrow\uparrow}) = B_{i+1} \subseteq \mathcal{C}(B) = B$, i.e. $B \in \mathrm{Int}(X, Y, I)$. Putting it together, $B \in \mathrm{Int}_C(X, Y, I)$. "⇐": Let $B \in \mathrm{Int}_C(X, Y, I)$. By definition, $B = C(B)$ and $B = B^{\downarrow\uparrow}$. Thus, $B_i = B$ ($i \in \mathbb{N}$), yielding $B = \mathcal{C}(B)$. ■

Using Theorem 3 one can compute C-intents and thus the complete lattice of C-concepts in case of finite $\mathbf{L}$ and Y. For instance, one can use a variant of Ganter's NextClosure [6] algorithm to list all fixed points of $\mathcal{C}$. We postpone details to a full version of this paper.

V. Selected Constraints

In this section we present several closure operators that can serve as examples of constraints. The operators can be further extended, combined together, etc. Each closure operator introduced in this section will be parameterized by additional values. Therefore, we will use the following convention for denoting particular instances of closure operators: each operator will be given a name written in capital letters (e.g. ADD, MODELOF, ...); list of parameters of an operator will be written in parenthesized list after its name (e.g. $\mathrm{ADD}(Z)$, $\mathrm{MODELOF}(T, *), \ldots$). In the sequel we assume that we work with a finite set Y of attributes, and $\mathbf{L}$ is a finite structure of truth degrees. Moreover, we assume the following convention for writing fuzzy sets. If $U = \{u_1, \ldots, u_n\}$ (finite universe) then a fuzzy set $A\colon U \to L$ will be denoted by $A = \{{}^{a_1}\!/u_1, \ldots, {}^{a_n}\!/u_n\}$ meaning that $A(u_i) = a_i$. For brevity, we write $\{\ldots, u, \ldots\}$ instead of $\{\ldots, {}^{1}\!/u, \ldots\}$, and we also omit elements of U whose membership degree is zero. Due to the limited scope of this paper, we postpone all proofs from this section to a full version of this paper.

$\mathrm{ADD}(Z)$, where $Z \in \mathbf{L}^Y$:
for each $B \in \mathbf{L}^Y$, we put $(\mathrm{ADD}(Z))(B) = B \cup Z$. That is, $\mathrm{ADD}(Z)$ is a fuzzy closure operator which determines intents containing Z. In other words, an intent B is $\mathrm{ADD}(Z)$-interesting iff each attribute from Y is contained in B at least to degree to which it belongs to Z. Notice that the boundary cases mentioned in Remark 2 (1) are given by choices $Z = \emptyset$ (empty $\mathbf{L}$-set, no constraint) and $Z = Y$ ($\mathbf{L}$-set containing each attribute in degree 1), respectively.

$\mathrm{CARDLEQ}(n)$, where n is a non-negative integer:
for each $B \in \mathbf{L}^Y$, define $(\mathrm{CARDLEQ}(n))(B)$ by

$$(\mathrm{CARDLEQ}(n))(B) = \begin{cases} B & \text{if } |B| \leq n, \\ Y & \text{otherwise,} \end{cases}$$

where $|B|$ denotes *cardinality of a fuzzy set* defined by

$$|B| = \textstyle\sum_{y\in Y} |\{a \in L \mid a < B(y)\}|. \quad (4)$$

Note that if $\mathbf{L}$ is a finite linear scale then (4) can be seen as representing the size of the (discrete) area below the graph of B. For instance, if the universe of $\mathbf{L}$ is $L = \{0, 0.25, 0.5, 0.75, 1\}$ with its genuine ordering, then $|\{{}^{0.75}\!/y, {}^{0.5}\!/z\}| = 3 + 2 = 5$. For $B = \emptyset$ (empty $\mathbf{L}$-set), we get $|B| = 0$; for $B = Y$, we have $|B| = |Y| \cdot (|L| - 1)$. Observe that by definition, $B \in \mathbf{L}^Y$ is $\mathrm{CARDLEQ}(n)$-interesting iff the cardinality of B is at most n or we have $B = Y$. Condition $B = Y$ is a technical one to ensure that $\mathrm{CARDLEQ}(n)$-interesting sets form a closure system. That is, $\mathrm{CARDLEQ}(n)$ can be used to determine intents up to a "certain number of fuzzy attributes". Let us stress, that "cardinalities of fuzzy sets" are usually defined by different formulas than that of (4). For instance, one can put $|B| = |\{y \in Y \mid B(y) > 0\}|$. If it is desirable, one may define $\mathrm{CARDLEQ}(n)$ using another notion of a cardinality. On the other hand, (4) has the following advantages: (i) it can be used with any finite residuated lattice taken as the structure of truth degrees and (ii) it respects partial membership degrees (for instance, for $L =$

$\{\dots, 0.5, 0.75, \dots\}$ being a finite subset of $[0,1]$ with its genuine ordering, we have $|\{^{0.5}/y\}| < |\{^{0.75}/y\}|$).

$\mathrm{SUPP}(n)$, where n is a non-negative integer:
for each $B \in \mathbf{L}^Y$, define $(\mathrm{SUPP}(n))(B)$ by

$$(\mathrm{SUPP}(n))(B) = \begin{cases} B & \text{if } |B^{\downarrow}| \geq n, \\ Y & \text{otherwise,} \end{cases}$$

where $|\cdots|$ is defined by (4). By definition, B is $\mathrm{SUPP}(n)$-interesting iff B equals Y or $|B^{\downarrow}| \geq n$. The latter condition means that cardinality of the fuzzy set of all objects sharing all attributes from B exceeds a user-defined parameter n. Note that in the ordinary setting, i.e. for $\mathbf{L}$ being the two-element Boolean algebra and for $|\cdots|$ being the classical notion of cardinality (of classical sets), $|B^{\downarrow}| \geq n$ means that at least n objects share all attributes from B—in terminology of association rules, the *support of* B is at least n [15]. In addition to that, in the classical setting, the collection of all $\mathrm{SUPP}(n)$-interesting intents (without Y) coincides with the set of closed frequent itemsets defined by Zaki [14] in order to get non-redundant association rules.

$\mathrm{CRISP}(Y')$, where $Y' \subseteq Y$:
for each $B \in \mathbf{L}^Y$, define $(\mathrm{CRISP}(Y'))(B)$ by

$$(\mathrm{CRISP}(Y'))(B) = \begin{cases} B & \text{if } B(y) \in \{0,1\} \ (y \in Y'), \\ Y & \text{otherwise.} \end{cases}$$

Hence, B is $\mathrm{CRISP}(Y')$-interesting iff each attributes from Y' is crisp in B, i.e. each $y \in Y'$ either belongs to B in degree 0 or in degree 1. Constraints on crispness of attributes are quite natural because sometimes it is desirable to work with fuzzy attributes and crisp attributes together. In such situations, attributes which are inherently crisp (e.g., "to have a driving license") should not belong to intents to other degrees than 0 or 1 (otherwise, concepts with such intents would probably be regarded as "not natural" or, worse yet, "confusing"). Using $\mathrm{CRISP}(Y')$, we can restrict ourselves only to concepts where attributes from Y' have crisp occurrences.

$\mathrm{EXCLUDE}(Z)$, where $Z \in \mathbf{L}^Y$:
for each $B \in \mathbf{L}^Y$, put

$$(\mathrm{EXCLUDE}(Z))(B) = \begin{cases} B & \text{if there is no } y \in Y \\ & \text{such that } 0 < Z(y) \leq B(y), \\ Y & \text{otherwise.} \end{cases}$$

That is, $B \in \mathbf{L}^Y$ is $\mathrm{EXCLUDE}(Z)$-interesting iff B equals Y (technical condition) or there is not any $y \in Y$ such that $Z(y) > 0$ and $Z(y) \leq B(y)$. In other words, $\mathbf{L}$-set B of attributes being $\mathrm{EXCLUDE}(Z)$-interesting means B *does not contain* any nonzero attribute from Z to a degree which exceeds the degree to which that attribute is contained in Z. Constraint based on EXCLUDE can be used to "disqualify (high) occurrences of attributes".

$\mathrm{SHARING}(A)$, where $Z \in \mathbf{L}^X$:
for each $B \in \mathbf{L}^Y$, we define $(\mathrm{SHARING}(A))(B)$ by

$$(\mathrm{SHARING}(A))(B) = \begin{cases} B & \text{if there is } x \in X \text{ such} \\ & \text{that } 0 < A(x) \leq B^{\downarrow}(x), \\ Y & \text{otherwise.} \end{cases}$$

Described verbally, B is $\mathrm{SHARING}(A)$-interesting iff B equals Y (technical condition) or at least one object in A is sharing all attributes from B. Since A is a fuzzy set of objects, a finer description of $(\mathrm{SHARING}(A))(B) = B$ is: "B equals Y or at least one object in X which belongs to A in a nonzero degree is sharing all attributes from B at least to degree prescribed by A". $\mathrm{SHARING}(A)$ is indeed a fuzzy closure operator which can be used to impose constraint on attributes to be "attributes of (at least one of) certain objects".

$\mathrm{HEDGE}(*)$, where $*$ is a (truth-stressing) hedge [8]:
for each $B \in \mathbf{L}^Y$, we put $(\mathrm{HEDGE}(*))(B) = B^{\downarrow * \uparrow}$. A detailed description of this constraint would require familiarity with result presented in [2]. We thus give only a sketch of the idea. Fuzzy concept lattices with hedges are a parameterized view on the original fuzzy concept lattice. The parameters involved are the so-called (truth-stressing) hedges. In general, a (truth-stressing) hedge $*$ is an additional unary operation on the complete residuated lattice $\mathbf{L}$ (our structure of truth degrees), i.e. it is a mapping $^* : L \to L$ satisfying, for each $a, b \in L$, $1^* = 1$, $a^* \leq a$, $(a \to b)^* \leq a^* \to b^*$, and $a^{**} = a^*$. Hedges are truth functions of logical connective "very true", see [7], [8]. Note that two boundary cases of hedges are (i) identity (no constraint), i.e. $a^* = a$ $(a \in L)$; (ii) globalization [13]: $1^* = 1$ and $a^* = 0$ $(a < 1)$. In formal concept analysis of data tables with fuzzy attributes, hedges are used as natural parameters of Galois connections which lead to reduction of the resulting structures: in [2] we showed that fuzzy concept lattices with hedges can be significantly smaller than the original ones. An important thing to stress if that from the point of view of the present results, a constraint given by a hedge $*$ in sense of [2] is but a particular constraint (namely, $\mathrm{HEDGE}(*)$) in sense of this paper.

$\mathrm{MODELOF}(T, *)$, where T is a set of fuzzy attribute implications [3], and $*$ is a (truth-stressing) hedge [8]. Before we specify $\mathrm{MODELOF}(T, *)$, let us note that *fuzzy attribute implications* are particular IF-THEN rules describing dependencies between fuzzy attributes, see [3]. A fuzzy attribute implication is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ are fuzzy sets of attributes. The intended meaning of $A \Rightarrow B$ is: "if an object has all attributes from A, then it has also all attributes from B". This meaning of fuzzy attribute implications can be formalized and one can also introduce motions of a model, a semantic entailment (parameterized by hedge $*$), a provability, etc., see [3]. From the point of view of constraints, a set T of fuzzy attribute implications can be seen as a set (supplied by a user) which describes IF-THEN dependencies between attributes. Thus, we may be interested in intents which respect (are models of) all fuzzy attributes implications in T. This constraint can be formalized in our setting as follows. For each $B \in \mathbf{L}^Y$ we define fuzzy sets of attributes [4]:

$$B_0 = B,$$
$$B_i = B_{i-1} \cup \bigcup \{D \otimes S(A, B_{i-1})^* \mid A \Rightarrow D \in T\},$$

and put $(\mathrm{MODELOF}(T, *))(B) = \bigcup_{i=0}^{\infty} B_i$. By definition, $(\mathrm{MODELOF}(T, *))(B)$ is the least model of T containing

	birth rate		death rate	
	low (bl)	high (bh)	low (dl)	high (dh)
Brazil (B)	0	0.25	0.25	0.5
Czech Republic (C)	0.75	0	0	0.75
Eritrea (E)	0	1	0	0.75
France (F)	0.5	0	0	0.5
Germany (G)	1	0	0	0.75
Iran (I)	0	0.25	0.5	0.25
Israel (L)	0	0.5	0.25	0.5
Japan (J)	0.75	0	0	0.5
Kenya (K)	0	1	0	1
Malaysia (M)	0	0.75	0.5	0.25
Poland (P)	0.75	0	0	0.75
Russia (R)	0.75	0	0	1
Singapore (S)	0.75	0	0.75	0.25
United States (U)	0	0.25	0	0.5
Venezuela (V)	0	0.5	0.5	0.25

Fig. 1. Illustrative data table with fuzzy attributes

B. Hence, B is $\text{MODELOF}(T,*)$-interesting iff B is a model of T (using $*$ as a parameter of the interpretation of fuzzy attribute implications [3]). Note that "MODELOF constraints" are in fact, the most general one, because each fuzzy closure operator on a finite Y can be completely described by a set of fuzzy attribute implications provided $*$ (hedge) is globalization [4] (we postpone details to the full version of this paper). Illustrative examples of "MODELOF constraints" will be given in Section VI.

VI. Examples of Constrained Lattices

In this section we present examples of constraints introduced in Section V which will be applied to a particular data table with fuzzy attributes describing properties of selected countries. First, let $\mathbf{L}$ (our structure of truth degrees) be a five-element Łukasiewicz chain with $L = \{0, 0.25, 0.5, 0.75, 1\}$. Thus, $\mathbf{L}$ is linearly ordered, $\wedge$ and $\vee$ coincide with minimum and maximum, respectively; $\otimes$ and $\rightarrow$ are defined by $a \otimes b = \max(a + b - 1, 0)$ and $a \rightarrow b = \min(1 - a + b, 1)$, see [1]. Consider now a data table with fuzzy attributes $\langle X, Y, I\rangle$ depicted in Fig. 1. The set X of objects consists of selected countries, the set Y of attributes consists of four attributes "low birth rate", "hight birth rate", "low death rate", and "hight death rate" describing birth/death rates in populations of the countries. Table entries indicate to which degree a given country has a low/high birth/death rate. The fuzzy concept lattice $\mathcal{B}(X, Y, I)$ generated from this data table contains 121 fuzzy concepts (clusters). The hierarchy of fuzzy concepts is depicted in Fig. 2. As one can see, the structure of $\mathcal{B}(X, Y, I)$ is large and hardly graspable for a user.

(1) Using $\text{CRISP}(\{\text{bl}, \text{bh}\})$, we can restrict ourselves only to concept in which the attributes concerning birth rates are crisp. Thus, we exclude concepts with birth rates other than 0 and 1. The resulting structure is depicted in Fig. 3. In Fig. 3 and further diagrams we use the following method of describing conceptual clusters (nodes of the diagram): when it is convenient, we use a verbal description of the fuzzy set of attributes which fall under the concept; objects are depicted by a color bar indicating degrees to which objects (represented by their abbreviations) fall under the concept (the darker the background color, the higher the degree; objects which belong to a concept in zero degree are not displayed).

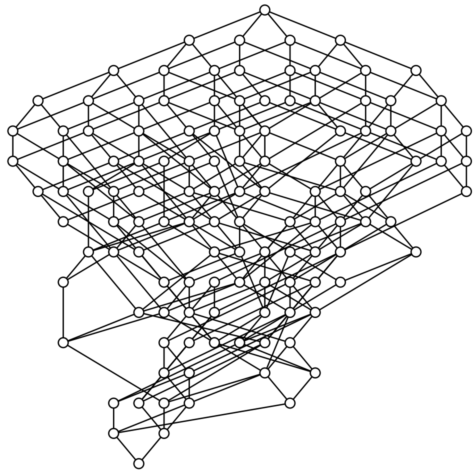

Fig. 2. Large and incomprehensible fuzzy concept lattice

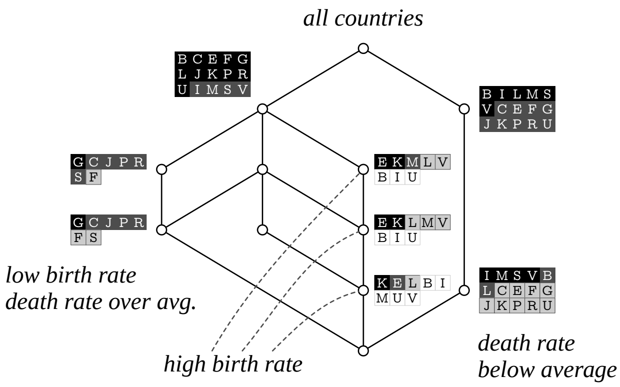

Fig. 3. Constrained fuzzy concept lattice (1)

(2) Constraints can be combined together in a conjunctive manner. That is, for fuzzy closure operators $C_1, \ldots, C_k$ representing constraints we can consider a fuzzy closure operator $C_1 \,\&\, \cdots \,\&\, C_k$ whose fixed points are fuzzy sets of attributes that are fixpoints of each C_i $(i = 1, \ldots, k)$. For instance, for $*$ (hedge) being globalization [13], we can consider a constraint given by

$$\text{SUPP}(4) \,\&\, \text{MODELOF}(\{\{{}^{0.25}\!/y\} \Rightarrow \{{}^{0.75}\!/y\} \mid y \in Y\}, *),$$

which says that a fuzzy set B of attributes is interesting iff its support is at lest 4 and, for each attribute y: $B(y)$ either is zero or $B(y) \geq 0.75$. This way we have restricted ourselves to concepts with support of a given size and with attributes which fall under the concept at least to a given threshold degree. The resulting structure is depicted in Fig. 4.

(3) Constraint $\text{EXCLUDE}(\{{}^{0.5}\!/\text{bl}\}) \,\&\, \text{ADD}(\{{}^{0.75}\!/\text{dh}\})$ delimits fuzzy concepts having "high death rate" at lest to degree 0.75 and having "low birth rate" at most to degree 0.25, see Fig. 5.

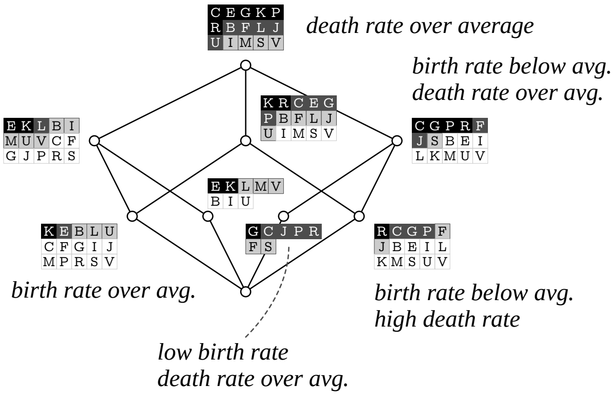

Fig. 4. Constrained fuzzy concept lattice (2)

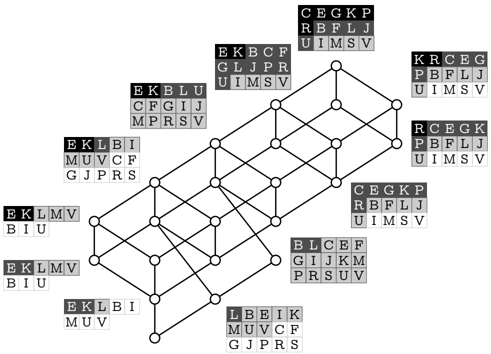

Fig. 5. Constrained fuzzy concept lattice (3)

(4) Consider the following composed operator

$$\mathrm{HEDGE}(*)\,\&\,\mathrm{SHARING}(\{\mathrm{B,E,K}\})\,\&\,$$
$$\mathrm{MODELOF}(\{\{{}^{0.25}\!/\mathrm{bl}\}\Rightarrow\{{}^{0.75}\!/\mathrm{bl}\},\{{}^{0.25}\!/\mathrm{bh}\}\Rightarrow\{\mathrm{bh}\},$$
$$\{{}^{0.25}\!/\mathrm{dl}\}\Rightarrow\{\mathrm{dl}\},\{{}^{0.25}\!/\mathrm{dh}\}\Rightarrow\{{}^{0.75}\!/\mathrm{dh}\}\},*).$$

If $*$ is globalization [13], then the operator represents a constraint which says that B is interesting iff (i) “Brazil”, “Eritrea”, or “Kenya” are sharing attributes from B; and (ii) B is a fuzzy set of attributes of a concept which satisfies the constraint given by globalization [2]; and (iii) B is a model of a given set of attributes (in detail, “bh” and “dl” must be crisp; attributes “bl” and “dh” belong to a concept either to zero degree or at lest to degree 0.75). The resulting structure is depicted in Fig. 6.

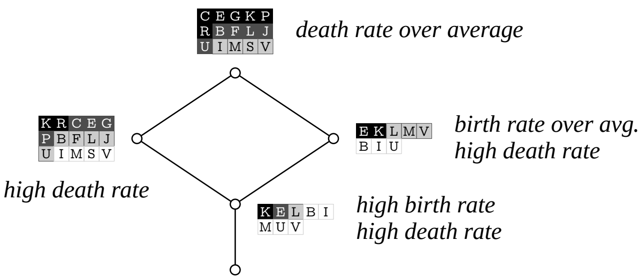

Fig. 6. Constrained fuzzy concept lattice (4)

VII. Further Issues

For limited scope of this paper, we did not present the following topics some of which will appear in a full paper or are subject of future research:

- Interactive specification of constraining fuzzy closure operators. An expert might not be able to explicitly describe a constraining fuzzy closure operator. However, he/she is usually able to tell which fuzzy concepts from the whole $\mathcal{B}(X,Y,I)$ are interesting. If $\mathcal{I}$ is a subset of $\mathcal{B}(X,Y,I)$ identified as (examples of) interesting concepts, an important problem is to describe a possibly largest fuzzy closure operator C such that each $\langle A,B\rangle \in \mathcal{I}$ is C-interesting (larger C means a better approximation of $\mathcal{I}$). The problem is to find a tractable description of C.
- Entailment of constraints. Intuitively, a constraint $\mathcal{C}_1$ (semantically) entails a constraint $\mathcal{C}_2$ iff each $B \in \mathbf{L}^Y$ satisfying $\mathcal{C}_1$ satisfies $\mathcal{C}_2$ as well. A study of entailment is important for obtaining small descriptions of constraining operators. Another problem that becomes interesting in a fuzzy setting is a graded entailment of constraints and related similarity issues (Do similar constraints lead to similar constrained structures?).
- More detailed results and more efficient algorithms for particular fuzzy closure operators can be obtained.
- Application of constraints formalized by closure operators to problems related with generating of non-redundant bases from data tables with fuzzy attributes [3].

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