Fuzzy attribute implications as formulas of logics with truth-evaluated syntax

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Abstract

We show that fuzzy attribute implications, which are the basic formulas of fuzzy attribute logic (FAL), can be introduced in several logics with truth-evaluated syntax. Namely, we focus on representing fuzzy attribute implications as formulas of threshold Boolean logic (TBL) and fuzzy Horn logic (FHL). Our motivations stem basically in the fact that TBL and FHL use formulas which are similar to that of FAL. This opens a question whether one can use results on TBL and FHL to get some insight into FAL, e.g. to prove the completeness of FAL. The present paper shows that in particular cases completeness of TBL and FHL can be used to prove completeness of FAL.

Keywords: Boolean logic, fuzzy logic, fuzzy attribute, if-then rule, threshold, provability degree, completeness

1. Introduction and preliminaries

In [2, 5, 6] we introduced fuzzy attribute implications as particular if-then rules and studied their properties. Fuzzy attribute implications are rules of the form "if A then B", where A and B are collections of fuzzy attributes (an attribute can apply to objects in degrees), with the meaning: if an object has all the attributes of A then it has also all attributes of B. Needles to say, if-then rules are perhaps the most frequently used means of knowledge representation and are used in data mining, expert systems, theory of control, etc. However, in real applications, we often face situations where we are compelled to use those rules in presence of vagueness (e.g., the behavior of a system cannot be described precisely due to its complexity). It is then quite natural to look at if-then rules from the viewpoint of fuzzy logic.

We are primarily interested in entailment of fuzzy attribute implications from sets of fuzzy attribute implications. That is, given a set of fuzzy attribute applications, we investigate to which degree a particular fuzzy attribute implication follows from that set. The semantic and syntactic entailments from sets of fuzzy attribute implications were studied in [5, 6]. In this paper we translate fuzzy attribute implications into formulas of other logics with truth-evaluated syntax and show that the existing results can be used to get some new characterizations of entailment from sets of fuzzy attribute implications.

Let us recall the preliminary notions. First, we need a suitable structure of truth degrees: we will use complete residuated lattice with truth-stressing hedges. A complete residuated lattice with a (truth-stressing) hedge is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a, b, c \in L$; hedge * satisfies (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [7, 8]. Two boundary cases of hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [10]: $a^* = 1$ if $a = 1$, $a^* = 0$ else.

Having $\mathbf{L}$ as our structure of truth degrees, we consider the following structural notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". $\{^a/u\}$ denotes the $\mathbf{L}$-set in U such that $\{^a/u\}(u) = a$, $\{^a/u\}(v) = 0$ for $v \neq u$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. We call $A \in \mathbf{L}^U$ finite if $\{u \in U \,|\, A(u) \neq 0\}$ is finite. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. The n-ary $\mathbf{L}$-relations on U can be thought of as $\mathbf{L}$-sets in U^n. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A,B) = \bigwedge_{u \in U}\bigl(A(u) \rightarrow B(u)\bigr)$, which generalizes the classical subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents the degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. Observe that $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [1, 7].

2. Fuzzy attribute logic

Fuzzy attribute logic is a logical calculus for reasoning with fuzzy attribute implications. *Fuzzy attribute implication* (*over attributes Y*) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has all attributes from B". For an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M by

$$||A \Rightarrow B||_M = S(A,M)^* \rightarrow S(B,M). \quad (1)$$

The role of a hedge * is (i) purely technical, see [2, 5]; (ii) * is a natural parameter of the interpretation of fuzzy attribute implications [3, 5]. Let T be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model* of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T)$. A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from T is defined by $||A \Rightarrow B||_T = \bigwedge_{M\in\mathrm{Mod}(T)} ||A \Rightarrow B||_M$. The following assertion presents two properties of the degree of semantic entailment $||\cdots||_T$ which will be used in the sequel.

Theorem 1 (i) $||A \Rightarrow B||_T = \bigwedge_{y\in Y} ||A \Rightarrow \{^{B(y)}/y\}||_T$*;*
(ii) $||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_T = 1\}$.

Proof. (i): For each $M \in \mathrm{Mod}(T)$ we have

$$\begin{aligned} ||A \Rightarrow B||_M &= S(A,M)^* \rightarrow S(B,M) = \\ &= \textstyle\bigwedge_{y\in Y} \big(S(A,M)^* \rightarrow (B(y) \rightarrow M(y))\big) = \\ &= \textstyle\bigwedge_{y\in Y} \big(S(A,M)^* \rightarrow S(\{^{B(y)}/y\},M)\big) = \\ &= \textstyle\bigwedge_{y\in Y} ||A \Rightarrow \{^{B(y)}/y\}||_M. \end{aligned}$$

Thus, $||A \Rightarrow B||_T = \bigwedge_{y\in Y} ||A \Rightarrow \{^{B(y)}/y\}||_T$.
(ii): See [6]. □

In [6] we introduced the *provability degree* $|\cdots|_T$ and proved the completeness of fuzzy attribute logic which says that $|\cdots|_T$ equals $||\cdots||_T$:

Theorem 2 (completeness of FAL [6]) *Let* $\mathbf{L}$ *and* Y *be finite. Then for every set* T *of fuzzy attribute implications and* $A \Rightarrow B$ *we have* $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$. □

3. Threshold Boolean logic

Threshold Boolean logic (TBL, see [11]) is a predicate calculus for reasoning with so-called threshold formulas which was originally developed to study the characterization of model classes of fuzzy structures by closure properties. TBL uses logical connectives with the classical interpretation and fuzzy structures as interpretations of languages. An analogous approach is described in [9]. In the sequel, we give a short survey of TBL.

By a *type* we mean a triplet $\langle R,F,\sigma\rangle$, where R is a set of relation symbols, F is a set of function symbols, $R\cap F = \emptyset$, and σ is a map assigning to each relation and function symbol $s \in F\cup R$ its arity $\sigma(s) \in \mathbb{N}_0$. Given finite $\mathbf{L}$, and type $\langle R,F,\sigma\rangle$, an $\mathbf{L}$*-language of type* $\langle R,F,\sigma\rangle$, denoted by $\mathcal{L}$, is a collection of symbols for logical connectives and quantifiers ($\Rightarrow$, $\neg$, and $\forall$), symbols for variables $x,y,\ldots$, relation symbols $r \in R$, function symbols $f \in F$, and auxiliary symbols (parentheses, etc.). In addition to that, we assume that $\mathcal{L}$ also contains all the elements of $\mathbf{L}$ which can be seen as *symbols for threshold truth degrees.* Further, we assume that each language contains binary relation symbol $\approx$ (symbol for equality). If $\mathbf{L}$ is clear from the context, we call $\mathbf{L}$-languages simply *languages*.

Terms (of language $\mathcal{L}$) are defined as usual. *Formulas of* $\mathcal{L}$ are defined as follows:

(i) if $t_1,\ldots,t_n$ are terms of $\mathcal{L}$, r is an n-ary relation symbol of $\mathcal{L}$, and $a \in L$, then $\langle r(t_1,\ldots,t_n),a\rangle$ is an (atomic) formula of $\mathcal{L}$,

(ii) if φ,ψ are formulas of $\mathcal{L}$ and x is a variable of $\mathcal{L}$, then $(\varphi \Rightarrow \psi)$, $\neg\varphi$, and $(\forall x)\varphi$ are formulas of $\mathcal{L}$,

In case of $\approx$, we write $\langle t \approx t',a\rangle$ instead of $\langle\approx(t,t'),a\rangle$. Formulas of TBL will be occasionally referred to as *threshold formulas*; formulas of this form are also known as *signed formulas* or *annotated formulas* [9]. TBL uses only $\Rightarrow$ and $\neg$ as the basic (symbols for) logical connectives and $\forall$ for the universal quantifier. The other logical connectives (conjunction, disjunction, etc.) and the existential quantifier can be introduced as shorthands for formulas, e.g., $\varphi \,\&\, \psi$ stands for $\neg(\varphi \Rightarrow \neg\psi)$, $\varphi \veebar \psi$ stands for $\neg\varphi \Rightarrow \psi$, etc. Further, we assume the usual conventions for writing formulas like the omission of parentheses.

In order to introduce the interpretation of threshold formulas, we need suitable semantic structures. Having a language $\mathcal{L}$ of type $\langle R,F,\sigma\rangle$ given, an $\mathbf{L}$*-structure for language* $\mathcal{L}$ is a triplet $\mathbf{M} = \langle M,R^{\mathbf{M}},F^{\mathbf{M}}\rangle$ such that $M \neq \emptyset$,

$$\begin{aligned} &R^{\mathbf{M}} = \{r^{\mathbf{M}}\colon M^n \rightarrow L \mid r \in R \text{ and } \sigma(r) = n\}, \\ &F^{\mathbf{M}} = \{f^{\mathbf{M}}\colon M^n \rightarrow M \mid f \in F \text{ and } \sigma(f) = n\}, \text{ and} \\ &u \approx^{\mathbf{M}} v = 1 \text{ iff } u = v \text{ for all } u,v \in M. \end{aligned}$$

That is, $\mathbf{M}$ represents a nonempty universe set M endowed with a collection $R^{\mathbf{M}}$ of $\mathbf{L}$-relations and a collection $F^{\mathbf{M}}$ of functions so that each n-ary relation symbol $r \in R$ corresponds with (is interpreted by) the n-ary $\mathbf{L}$-relation $r^{\mathbf{M}} \in R^{\mathbf{M}}$ and each n-ary function symbol $f \in F$ corresponds with (is interpreted by) the n-ary function $f^{\mathbf{M}} \in F^{\mathbf{M}}$. If $\mathbf{L}$ is the two-element Boolean algebra then the $\mathbf{L}$-structures are but the classical structures of predicate logic.

An $\mathbf{M}$-*valuation* is a mapping v assigning to each variable x an element of M. For $\mathbf{M}$-valuations v,w we write $v \equiv_x w$ if $v(y) = w(y)$ for each variable y distinct from x. If $\mathbf{M}$ is clear from the context, $\mathbf{M}$-valuations shall be simply called valuations. We let $||t||_{\mathbf{M},v}$ denote the interpretation of the term t in $\mathbf{M}$ under v. φ *is true in* $\mathbf{M}$ *under* v, written $\mathbf{M} \models \varphi[v]$, if

(i) φ is $\langle r(\cdots t_i \cdots),a\rangle$, and $r^{\mathbf{M}}(\cdots ||t_i||_{\mathbf{M},v} \cdots) \geq a$;

(ii) φ is $\psi \Rightarrow \chi$, and if $\mathbf{M} \models \psi[v]$ then $\mathbf{M} \models \chi[v]$;

(iii) φ is $\neg\psi$, and $\mathbf{M} \not\models \psi[v]$;
(iv) φ is $(\forall x)\psi$, and $\mathbf{M} \models \psi[w]$ for each $w \equiv_x v$.

φ *holds* (*is valid*) *in* $\mathbf{M}$ if $\mathbf{M} \models \varphi[v]$ for each $\mathbf{M}$-valuation v. Observe that the definition of $\mathbf{M} \models \varphi[v]$ distinguishes from the classical one only in (i), because TBL uses atomic formulas with threshold degrees. That is, TBL uses logical connectives $\Rightarrow$ and $\neg$ with the classical interpretation.

Finally, we need notions of a model and a semantic entailment. Given a set T of formulas of $\mathcal{L}$, an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathcal{L}$ is called a *model of* T, written $\mathbf{M} \in \mathrm{Mod}(T)$, if each $\varphi \in T$ holds in $\mathbf{M}$. φ *semantically follows from* T, denoted by $T \models \varphi$, if φ holds in each model of T. In [11] we also introduced the notion of *provability*; the fact that φ is provable from T is denoted by $T \vdash \varphi$. Due to the limited scope of this paper we do not describe $\vdash$; it is not even necessary to understand the transformation presented below because the transformation is a model-theoretical one. Note that the completeness theorem of TBL says that the syntactic entailment $\vdash$ (provability) coincides with the semantic entailment $\models$:

Theorem 3 (completeness of TBL [11]) *Let T be a set of formulas of $\mathcal{L}$. Then for each φ of $\mathcal{L}$ we have $T \vdash \varphi$ iff $T \models \varphi$.* □

Completeness of FAL via TBL

Since TBL is a predicate calculus, we first need to specify a language and a way to translate implications with fuzzy attributes into threshold formulas of that language. So, let $Y = \{y_1,\dots,y_n\}$ be a set of attributes. Consider the language $\mathcal{L}$ of type $\langle R,F,\sigma\rangle$, where $R = \{r,\approx\}$, $F = \{c_1,\dots,c_n\}$, $\sigma(r) = 1$, and $\sigma(c_1) = \cdots = \sigma(c_n) = 0$. Thus, r is a unary relation symbol and $c_1,\dots,c_n$ are constants. For each fuzzy attribute implication $A \Rightarrow B$ over Y we consider a formula $\mathrm{thr}(A \Rightarrow B)$ of $\mathcal{L}$ defined by

$$(\langle r(c_1),A(y_1)\rangle \,\&\cdots\&\langle r(c_n),A(y_n)\rangle) \Rightarrow \\ \Rightarrow (\langle r(c_1),B(y_1)\rangle \,\&\cdots\&\langle r(c_n),B(y_n)\rangle).$$

Now for each set T of fuzzy attribute implications over Y we define a set $\mathrm{thr}(T)$ of formulas of $\mathcal{L}$ as follows:

$$\begin{aligned}\mathrm{thr}(T) = \{&\mathrm{thr}(A \Rightarrow B)\,|\,A \Rightarrow B \in T\} \cup \\ \{&\neg\langle c_i \approx c_j, 1\rangle\,|\,i \neq j\} \cup \\ \{&(\forall x)(\langle x \approx c_1, 1\rangle \veebar \cdots \veebar \langle x \approx c_n, 1\rangle)\}.\end{aligned}$$

Further, for each $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes we consider an $\mathbf{L}$-structure $\mathrm{thr}(M) = \langle Y, R^{\mathrm{thr}(M)}, F^{\mathrm{thr}(M)}\rangle$ for $\mathcal{L}$ such that $y \approx^{\mathrm{thr}(M)} y = 1$ ($y \in Y$), $y_i \approx^{\mathrm{thr}(M)} y_j = 0$ ($i \neq j$), $r^{\mathrm{thr}(M)} = M$, and $c_i^{\mathrm{thr}(M)} = y_i$ ($i = 1,\dots,n$). The following assertion characterizes the relationship between M and $\mathrm{thr}(M)$ from the viewpoint of validity.

Lemma 4 *Let $\mathbf{L}$ be a residuated lattice with globalization. Then $||A \Rightarrow B||_M = 1$ iff $\mathrm{thr}(M) \models \mathrm{thr}(A \Rightarrow B)$.*

Proof. Observe that $||A \Rightarrow B||_M = 1$ iff $A \subseteq M$ implies $B \subseteq M$ iff $A \subseteq r^{\mathrm{thr}(M)}$ implies $B \subseteq r^{\mathrm{thr}(M)}$ iff

$$A(y_i) \leq r^{\mathrm{thr}(M)}\big(c_i^{\mathrm{thr}(M)}\big) \quad (i = 1,\dots,n)$$

implies

$$B(y_i) \leq r^{\mathrm{thr}(M)}\big(c_i^{\mathrm{thr}(M)}\big) \quad (i = 1,\dots,n),$$

which holds true iff

$$\mathrm{thr}(M) \models (\langle r(c_1),A(y_1)\rangle \,\&\cdots\&\langle r(c_n),A(y_n)\rangle)$$

implies

$$\mathrm{thr}(M) \models (\langle r(c_1),B(y_1)\rangle \,\&\cdots\&\langle r(c_n),B(y_n)\rangle),$$

which is true iff $\mathrm{thr}(M) \models \mathrm{thr}(A \Rightarrow B)$. □

The following theorem establishes the connection between FAL and TBL. Due to the semantics of TBL, the assertion is limited only to the globalization taken as the truth-stressing hedge.

Theorem 5 (completeness of FAL via TBL) *Let $\mathbf{L}$ be a finite residuated lattice with globalization. Then for each set T of fuzzy attribute implications we have*

$$||A \Rightarrow B||_T = \bigvee\{c \in L\,|\,\mathrm{thr}(T) \vdash \mathrm{thr}(A \Rightarrow c \otimes B)\}.$$

Proof. First, observe that if M is a model of T, then $\mathrm{thr}(M)$ is a model of $\mathrm{thr}(T)$. Conversely, if $\mathbf{M}$ is a model of $\mathrm{thr}(T)$, then $M \in \mathbf{L}^Y$ defined by $M(y_i) = r^{\mathbf{M}}(c_i^{\mathbf{M}})$ ($i = 1,\dots,n$) is a model of T such that $\mathrm{thr}(M) \models \mathrm{thr}(A \Rightarrow B)$ iff $\mathbf{M} \models \mathrm{thr}(A \Rightarrow B)$. Therefore, Lemma 4 gives $||A \Rightarrow B||_T = 1$ iff $\mathrm{thr}(T) \models \mathrm{thr}(A \Rightarrow B)$ which together with Theorem 1 (ii) and Theorem 3 yields

$$\begin{aligned}||A \Rightarrow B||_T &= \bigvee\{c \in L\,|\,||A \Rightarrow c \otimes B||_T = 1\} = \\ &= \bigvee\{c \in L\,|\,\mathrm{thr}(T) \models \mathrm{thr}(A \Rightarrow c \otimes B)\} = \\ &= \bigvee\{c \in L\,|\,\mathrm{thr}(T) \vdash \mathrm{thr}(A \Rightarrow c \otimes B)\}.\end{aligned}$$ □

4. Fuzzy Horn logic

Fuzzy Horn logic (FHL) was introduced in [3]. This calculus was developed to deal with equalities in fuzzy setting (FHL is Pavelka-style complete [7], model classes of FHL are characterized by closure properties). Unlike TBL, FHL interprets logical connectives by operations of general residuated lattices and defines a notion of a degree of provability. Languages of FHL consist of function symbols and a single relation symbol $\approx$ for equality. Thus, the translation of fuzzy attribute implications into formulas of FHL (so-called Horn clauses with truth-weighted premises) will be more complicated because we can use the only relation symbol $\approx$ which is interpreted by an $\mathbf{L}$-equality relation (see [3, 4]).

Assume we are given a language $\mathcal{L}$ of type $\langle R,F,\sigma\rangle$, where $R = \{\approx\}$. A *Horn clause* (*of* $\mathcal{L}$) is an expression

$P \Rightarrow t \approx t'$, where t, t' are terms of $\mathcal{L}$, and P is a finite binary **L**-relation on the set of all terms of $\mathcal{L}$. Let $P \Rightarrow t \approx t'$ be a Horn clause of $\mathcal{L}$. Given an **L**-structure for $\mathcal{L}$ and an **M**-valuation v, we define the *degree* $||P \Rightarrow t \approx t'||_{\mathbf{M},v}$ *to which* $P \Rightarrow t \approx t'$ *is true in* **M** *under* v:

$$||P \Rightarrow t \approx t'||_{\mathbf{M},v} = S(P, \theta_{v^\sharp})^* \to ||t||_{\mathbf{M},v} \approx^{\mathbf{M}} ||t'||_{\mathbf{M},v},$$

where $\theta_{v^\sharp}(s,s') = ||s||_{\mathbf{M},v} \approx^{\mathbf{M}} ||s'||_{\mathbf{M},v}$ for all terms s, s' of $\mathcal{L}$. The *degree* $||P \Rightarrow t \approx t'||_{\mathbf{M}}$ *to which* $P \Rightarrow t \approx t'$ *is true in* **M** is defined by

$$||P \Rightarrow t \approx t'||_{\mathbf{M}} = \bigwedge\{||P \Rightarrow t \approx t'||_{\mathbf{M},v} \,|\, v \text{ is a valuation}\}.$$

L-sets of Horn clauses (of the same language) will be denoted by $\Sigma, \Sigma', \ldots$, $\Sigma(P \Rightarrow t \approx t')$ being interpreted as "the degree to which $P \Rightarrow t \approx t'$ belongs to Σ". An **L**-structure **M** for $\mathcal{L}$ is called a *model of* Σ, written $\mathbf{M} \in \mathrm{Mod}(\Sigma)$, if **M** satisfies (i) $u \approx^{\mathbf{M}} v = v \approx^{\mathbf{M}} u$, (ii) $u \approx^{\mathbf{M}} v \otimes v \approx^{\mathbf{M}} w \leq u \approx^{\mathbf{M}} w$, (iii) $u_1 \approx^{\mathbf{M}} v_1 \otimes \cdots \otimes u_n \approx^{\mathbf{M}} v_n \leq f^{\mathbf{M}}(u_1, \ldots, u_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(v_1, \ldots, v_n)$, and $\Sigma(P \Rightarrow t \approx t') \leq ||P \Rightarrow t \approx t'||_{\mathbf{M}}$ for each Horn clause $P \Rightarrow t \approx t'$. The *degree* $||P \Rightarrow t \approx t'||_{\Sigma}$ *to which* $P \Rightarrow t \approx t'$ *semantically follows from* Σ is defined by

$$||P \Rightarrow t \approx t'||_{\Sigma} = \bigwedge\{||P \Rightarrow t \approx t'||_{\mathbf{M}} \,|\, \mathbf{M} \in \mathrm{Mod}(\Sigma)\}.$$

FHL defines degrees $|\cdots|_{\Sigma} \in L$ of provability of Horn clauses from **L**-sets of Horn clauses [3]. If **L** is a finite residuated lattice with a Horn hedge [3], FHL is Pavelka-style complete: $|P \Rightarrow t \approx t'|_{\Sigma} = ||P \Rightarrow t \approx t'||_{\Sigma}$ for any Σ and $P \Rightarrow t \approx t'$. Again, we do not describe $|\cdots|_{\Sigma}$ in more detail since this is not necessary for the subsequent transformation.

Completeness of FAL via FHL

Due to the limited scope of this paper, we present only a sketch of the procedure. For a set $Y = \{y_1, \ldots, y_n\}$ of attributes we let $\mathcal{L}$ be the language of type $\langle R, F, \sigma \rangle$, where $F = \{c_1, \bar{c}_1, \ldots, c_n, \bar{c}_n\}$ is a set of constants and $R = \{\approx\}$. For each **L**-set $A \in \mathbf{L}^Y$ and $y = y_i \in Y$ we consider Horn clause $P \Rightarrow \bar{c}_i \approx c_i$, abbreviated $\mathrm{fhl}(A, y)$, where $P(\bar{c}_j, c_j) = A(y_j)$ $(j = 1, \ldots, n)$, $P(s, s') = 0$ else. Further, for a set T of fuzzy attribute implications over Y we define an **L**-set $\mathrm{fhl}(T)$ of Horn clauses by $(\mathrm{fhl}(T))(\mathrm{fhl}(A, y)) = \bigvee\{B(y) \,|\, A \Rightarrow B \in T\}$, $(\mathrm{fhl}(T))(\varphi) = 0$ else. Finally, for each $M \in \mathbf{L}^Y$ we consider an **L**-structure $\mathrm{fhl}(M) = \langle Y \cup \{\bar{y}_1, \ldots, \bar{y}_n\}, R^{\mathrm{fhl}(M)}, F^{\mathrm{fhl}(M)} \rangle$ such that $\bar{c}_i^{\mathrm{fhl}(M)} = \bar{y}_i$,

$$c_i^{\mathrm{fhl}(M)} = \begin{cases} y_i & \text{if } M(y_i) \neq 1, \\ \bar{y}_i & \text{otherwise,} \end{cases} \qquad (i = 1, \ldots, n),$$

and $\approx^{\mathrm{fhl}(M)}$ is the least **L**-equality on $Y \cup \{\bar{y}_1, \ldots, \bar{y}_n\}$ such that $\bar{c}_i^{\mathrm{fhl}(M)} \approx^{\mathrm{fhl}(M)} c_i^{\mathrm{fhl}(M)} = M(y_i)$ $(i = 1, \ldots, n)$.

Lemma 6 (i) $||A \Rightarrow \{{}^1/y\}||_M = ||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(M)}$;
(ii) *if* $M \in \mathbf{L}^Y$ *is a model of* T *then* $\mathrm{fhl}(M) \in \mathrm{Mod}(\mathrm{fhl}(T))$;
(iii) *if* **M** *is a model of* $\mathrm{fhl}(T)$ *then* $M \in \mathbf{L}^Y$ *defined by* $M(y_i) = \bar{c}_i^{\mathbf{M}} \approx^{\mathbf{M}} c_i^{\mathbf{M}}$ $(i = 1, \ldots, n)$ *is a model of* T *such that* $||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(M)} = ||\mathrm{fhl}(A, y)||_{\mathbf{M}}$ $(A \in \mathbf{L}^Y, y \in Y)$. □

Theorem 7 (completeness of FAL via FHL) *Let* **L** *be a finite residuated lattice with hedge for which FHL is complete. Then for each set* T *of fuzzy attribute implications we have* $||A \Rightarrow B||_T = S(B, \mathrm{fhl}_T(A))$*, where* $\mathrm{fhl}_T(A) \in \mathbf{L}^Y$ *is defined by* $(\mathrm{fhl}_T(A))(y) = |\mathrm{fhl}(A, y)|_{\mathrm{fhl}(T)}$.

Proof. Lemma 6 yields that $||A \Rightarrow \{{}^b/y\}||_T = 1$ holds iff $b \leq ||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(T)}$. Thus, using Theorem 1, Lemma 6, and Pavelka-style completeness of FHL, we get

$$\begin{aligned}
&||A \Rightarrow B||_T = \bigvee\{c \in L \,|\, ||A \Rightarrow c \otimes B||_T = 1\} = \\
&= \bigvee\{c \in L \,|\, ||A \Rightarrow \{{}^{c \otimes B(y)}/y\}||_T = 1 \text{ for each } y\} = \\
&= \bigvee\{c \in L \,|\, c \otimes B(y) \leq ||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(T)} \text{ for each } y\} = \\
&= \bigvee\{c \in L \,|\, c \leq \textstyle\bigwedge_{y \in Y}\big(B(y) \to ||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(T)}\big)\} = \\
&= \textstyle\bigwedge_{y \in Y}\big(B(y) \to ||\mathrm{fhl}(A, y)||_{\mathrm{fhl}(T)}\big) = \\
&= \textstyle\bigwedge_{y \in Y}\big(B(y) \to |\mathrm{fhl}(A, y)|_{\mathrm{fhl}(T)}\big) = S(B, \mathrm{fhl}_T(A)). \quad \square
\end{aligned}$$

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