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Axiomatizations of Fuzzy Attribute Logic*

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Abstract. We study fuzzy attribute logic, i.e. a logic for reasoning about formulas of the form $A \Rightarrow B$ where A and B are fuzzy sets (non-sharp collections) of attributes. A formula $A \Rightarrow B$ is true in a data table with fuzzy attributes iff each object having all attributes from A has also all attributes from B, membership degrees of A and B playing a role of thresholds. We present a set of axioms and prove syntactico-semantical completeness with respect to the data table semantics. We also prove some derived rules in our axiomatic system. Furthermore, we introduce a notion of a degree to which a fuzzy set T of formulas entails a formula $A \Rightarrow B$ and prove completeness in Pavelka style (graded completeness) which says that a degree to which $A \Rightarrow B$ semantically follows from T equals a degree to which $A \Rightarrow B$ is provable from T.

1 Introduction

If-then rules are perhaps the most common way to express our knowledge. Basically, rules are extracted from data to bring up a new knowledge about the data or are formulated by a user/expert to represent a constraint on the data. Rules of the form $A \Rightarrow B$, where A and B are collections of attributes, have been used in data mining and in databases. In data mining, rules $A \Rightarrow B$ are called association rules, see e.g. [28], or attribute implications in formal concept analysis, see [15, 19], and have the following basic meaning: If an object has all attributes from A then it has all attributes from B. The goal then is to extract "all interesting rules" from data. In databases, rules $A \Rightarrow B$ are called functional dependencies, see [24] for a good overview, and have the following basic meaning: If any two objects (items, rows) of a database agree in their values on each of the attributes from A then they agree on each attribute from B.

Armstrong [1] introduced a set of inference rules, a modified version of which became known as Armstrong axioms. Armstrong axioms are complete w.r.t. the above-described database semantics. That is, a $A \Rightarrow B$ is provable (using Armstrong axioms) from a set T of rules if and only if $A \Rightarrow B$ semantically follows from T, i.e. if every database satisfying each rule from T satisfies also

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$A \Rightarrow B$. It is also known that Armstrong axioms are complete w.r.t. the above-described object-attribute semantics [15, Section 2.3].

We are interested in rules $A \Rightarrow B$ and their object-attribute semantics from the point of view of fuzzy logic. Following [15], we call these rules *attribute implications*. Attribute implications in a fuzzy setting were first considered by Pollandt [26]. In a series of papers [6, 8, 10, 11], we investigated attribute implications from the point of view of a rather general fuzzy logic which covers most of the particular cases used in applications: both finite and infinite scales of truth degrees, various fuzzy logical conjunctions, etc. [23, 20]. We defined an object-attribute semantics, i.e. attribute implications are interpreted in data tables with rows and columns corresponding to objects and attributes, respectively, and table entries describing degrees to which objects have attributes. Furthermore, we described non-redundant bases of attribute implications, i.e. minimal sets of attribute implications entailing all true implications of a given data table, and an algorithm with a polynomial time delay [22] to generate non-redundant bases.

The present paper is a follow-up to [11]. We introduce a fuzzy attribute logic, a logic for reasoning with (fuzzy) attribute implications in a fuzzy setting. We are interested in completeness w.r.t. object-attribute semantics in two ways. First, the usual completeness saying that for a set T of (fuzzy) attribute implications and a (fuzzy) attribute implication $A \Rightarrow B$, $A \Rightarrow B$ is provable from T if and only if $A \Rightarrow B$ semantically follows from T. Second, in graded completeness (Pavelka-style completeness) [16, 25] saying that for a fuzzy set T of (fuzzy) attribute implications and a (fuzzy) attribute implication $A \Rightarrow B$, a degree to which $A \Rightarrow B$ is provable from T equals a degree to which $A \Rightarrow B$ semantically follows from T. We show that axioms for fuzzy attribute logic can be formulated as consisting of two groups: First, Armstrong-like axioms capturing the structure of reasoning with rules. Second, axioms capturing reasoning with truth degrees in rules. Also, we comment on related approaches and future research.

2 Preliminaries

Fuzzy logic and fuzzy set theory are formal frameworks for a manipulation of a particular form of imperfection called fuzziness (vagueness). Contrary to classical logic, fuzzy logic uses a scale L of truth degrees, a most common choice being $L = [0, 1]$ (real unit interval) or some subchain of $[0, 1]$. This enables to consider intermediate truth degrees of propositions, e.g. "object x has attribute y" has a truth degree 0.8 indicating that the proposition is almost true. In addition to a set L of truth degrees, one has to pick an appropriate collection of logical connectives (implication, conjunction, ...). A general choice of a set of truth degrees plus logical connectives is represented by so-called complete residuated lattices (equipped possibly with additional operations). The rest of this section presents an introduction to fuzzy logic notions we need in the sequel. Details can be found e.g. in [3, 18, 20], a good introduction to fuzzy logic and fuzzy sets is presented in [23].

A complete residuated lattice with a truth-stressing hedge (shortly, a hedge) [20, 21] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \tag{2}$$

$$a^* \leq a, \tag{3}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{4}$$

$$a^{**} = a^*, \tag{5}$$

for each $a, b \in L$, $a_i \in L$ $(i \in I)$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [20, 21]. Properties (3)–(5) have natural interpretations, e.g. (3) can be read: "if a is very true, then a is true", (4) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{6}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 \text{ if } a \leq b, \\ b \text{ otherwise}, \end{cases} \end{aligned} \tag{7}$$

$$\text{Goguen (product):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 \;\text{ if } a \leq b, \\ \frac{b}{a} \text{ otherwise}. \end{cases} \end{aligned} \tag{8}$$

In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ $(a_0 < \cdots < a_n)$ with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0, 1]$ to L.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ $(a \in L)$; (ii) globalization [27]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{9}$$

A special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle\{0,1\},\wedge,\vee,\otimes,\rightarrow,{}^*,0,1\rangle$, denoted by **2**, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge,\vee,\otimes,\rightarrow$ of **2** are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$. Note that if we prove an assertion for general **L**, then, in particular, we obtain a "crisp version" of this assertion for **L** being **2**.

Having **L**, we define usual notions: an **L**-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1,\ldots,u_n\}$ then A can be denoted by $A = \{{}^{a_1}/u_1,\ldots,{}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1,\ldots,n$. For brevity, we introduce the following convention: we write $\{\ldots,u,\ldots\}$ instead of $\{\ldots,{}^{1}/u,\ldots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u,{}^{0.5}/v\}$ instead of $\{{}^{1}/u,{}^{0.5}/v,{}^{0}/w\}$, etc. Let $\mathbf{L}^U$ denote the collection of all **L**-sets in U. The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A,B \in \mathbf{L}^U$ is an **L**-set $A \cap B$ in U such that $(A\cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in \mathbf{L}^{X\times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ (a degree to which x and y are related by I). An **L**-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0,1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, $a \otimes A \in \mathbf{L}^X$ is defined by $(a \otimes A)(x) = a \otimes A(x)$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A,B) = \textstyle\bigwedge_{u\in U}\bigl(A(u) \rightarrow B(u)\bigr), \tag{10}$$

which generalizes the classical subsethood relation $\subseteq$. $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [3, 20]. Throughout the rest of the paper, **L** denotes an arbitrary complete residuated lattice with a hedge.

3 Fuzzy attribute logic

3.1 Attribute implications and their validity

We first introduce attribute implications. Suppose Y is a finite set of attributes. A (*fuzzy*) *attribute implication* (*over attributes* Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). Fuzzy attribute implications are the basic formulas of fuzzy attribute logic.

The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B" with the logical connectives being given by $\mathbf{L}$. A fuzzy attribute implication does not have any kind of "validity" on its own—it is a syntactic notion.

Remark 1. For an fuzzy attribute implication $A \Rightarrow B$, both A and B are fuzzy sets of attributes. Particularly, A and B can both be ordinary sets (i.e. $A(y) \in \{0, 1\}$ and $B(y) \in \{0, 1\}$ for each $y \in Y$), i.e. ordinary attribute implications (association rules, functional dependencies) are a special case of fuzzy attribute implications.

In order to consider validity, we must introduce an interpretation of fuzzy attribute implications. Fuzzy attribute implications are meant to be interpreted in data tables with fuzzy attributes [6, 8, 10]. A data table with fuzzy attributes can be seen as a triplet $\mathcal{T} = \langle X, Y, I\rangle$ where X is a set of objects, Y is a finite set of attributes (the same as above in the definition of a fuzzy attribute implication), and $I \in \mathbf{L}^{X\times Y}$ is a binary $\mathbf{L}$-relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x, y)$ to which x has y. $\mathcal{T} = \langle X, Y, I\rangle$ can be thought as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$. A row of a table $\mathcal{T} = \langle X, Y, I\rangle$ corresponding to an object $x \in X$ can be seen as a fuzzy set I_x of attributes to which an attribute $y \in Y$ belongs to a degree $I_x(y) = I(x, y)$. Forgetting now for a while about the data table, any fuzzy set $M \in \mathbf{L}^Y$ can be seen as a fuzzy set of attributes of some object with $M(y)$ being a degree to which the object has attribute y. For fuzzy set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M by

$$||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M). \tag{11}$$

It is easily seen that if M is a fuzzy set of attributes of some object x then $||A \Rightarrow B||_M$ is the degree to which "if it is (very) true that x has all attributes from A then x has all attributes from B". For a system $\mathcal{M}$ of $\mathbf{L}$-sets in Y, define a degree $||A \Rightarrow B||_{\mathcal{M}}$ to which $A \Rightarrow B$ is true in (each M from) $\mathcal{M}$ by

$$||A \Rightarrow B||_{\mathcal{M}} = \bigwedge_{M\in\mathcal{M}} ||A \Rightarrow B||_M. \tag{12}$$

Finally, given a data table $\mathcal{T} = \langle X, Y, I\rangle$ and putting $\mathcal{M} = \{I_x \,|\, x \in X\}$, $||A \Rightarrow B||_{\mathcal{M}}$ is a degree to which it is true that $A \Rightarrow B$ is true in each row of table $\mathcal{T}$, i.e. a degree to which "for each object $x \in X$: if it is (very) true that x has all attributes from A, then x has all attributes from B". This degree is denoted by $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ and is called a degree to which $A \Rightarrow B$ is true in data table $\langle X, Y, I\rangle$.

Remark 2. For a fuzzy attribute implication $A \Rightarrow B$, degrees $A(y) \in L$ and $B(y) \in L$ can be seen as thresholds. This is best seen when * is globalization, i.e. $1^* = 1$ and $a^* = 0$ for $a < 1$. Since for $a, b \in L$ we have $a \leq b$ iff $a \rightarrow b = 1$, we have

$$(a \rightarrow b)^* = \begin{cases} 1 & \text{iff } a \leq b, \\ 0 & \text{iff } a \not\leq b. \end{cases}$$

Table 1. Data table with fuzzy atributes

I	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1.0	1.0	0.0	1.0	1.0	0.2
x_2	1.0	0.4	0.3	0.8	0.5	1.0
x_3	0.2	0.9	0.7	0.5	1.0	0.6
x_4	1.0	1.0	0.8	1.0	1.0	0.5

$X = \{x_1, \ldots, x_4\}$

$Y = \{y_1, \ldots, y_6\}$

Therefore, $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ means that a proposition "for each object $x \in X$: if for each attribute $y \in Y$, x has y in degree greater than or equal to (a threshold) $A(y)$, then for each $y \in Y$, x has y in degree at least $B(y)$" has a truth degree 1 (is fully true). In general, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is a truth degree of the latter proposition. As a particular example, if $A(y) = a$ for $y \in Y_A \subseteq Y$ (and $A(y) = 0$ for $y \notin Y_A$) $B(y) = b$ for $y \in Y_B \subseteq Y$ (and $B(y) = 0$ for $y \notin Y_B$), the proposition says "for each object $x \in X$: if x has all attributes from Y_A in degree at least a, then x has all attributes from Y_B in degree at least b", etc. That is, having A and B fuzzy sets allows for a rich expressibility of relationships between attributes which is why we want A and B to be fuzzy sets in general.

Example 1. For illustration, consider Tab. 3.1, where table entries are taken from **L** defined on the real unit interval $L = [0, 1]$ with globalization. Consider now the following fuzzy attribute implications.

(1) $\{{}^{0.3}\!/y_3, {}^{0.7}\!/y_4\} \Rightarrow \{y_1, {}^{0.3}\!/y_2, {}^{0.8}\!/y_4, {}^{0.4}\!/y_6\}$ is true in degree 1 in data table from Tab. 3.1. On the other hand, implication $\{y_1, {}^{0.3}\!/y_3\} \Rightarrow \{{}^{0.1}\!/y_2, {}^{0.7}\!/y_5, {}^{0.4}\!/y_6\}$ is not true in degree 1 in Tab. 3.1—object x_2 can be taken as a counterexample: x_2 does not have attribute y_5 in degree greater than or equal to 0.7.

(2) $\{y_1, y_2\} \Rightarrow \{y_4, y_5\}$ is a crisp attribute implication which is true in degree 1 in the table. On the contrary, $\{y_5\} \Rightarrow \{y_4\}$ is also crisp but it is not true in degree 1 (object x_3 is a counterexample).

(3) Implication $\{{}^{0.5}\!/y_5, {}^{0.5}\!/y_6\} \Rightarrow \{{}^{0.3}\!/y_2, {}^{0.3}\!/y_3\}$ is in the above-mentioned form for $Y_A = \{y_5, y_6\}$, $Y_B = \{y_2, y_3\}$, $a = 0.5$, and $b = 0.3$. The implication is true in data table in degree 1. $\{{}^{0.5}\!/y_5, {}^{0.5}\!/y_6\} \Rightarrow \{{}^{0.3}\!/y_1, {}^{0.3}\!/y_2\}$ is also in this form (for $Y_B = \{y_1, y_2\}$) but it is not true in the data table in degree 1 (again, take x_3 as a counterexample).

3.2 Semantic entailment and further semantic notions

Consider an **L**-set T of fuzzy attribute implications. From the point of view of logic, T can be seen as a theory, i.e. a degree $T(A \Rightarrow B)$ to which $A \Rightarrow B$ belongs to T can be seen as a degree to which we assume the validity of $A \Rightarrow B$. This corresponds to the notion of a theory as a fuzzy set of axioms in fuzzy logic [16, 25]. From the user's point of view, T can be seen a fuzzy set of implications extracted from data such that $T(A \Rightarrow B)$ is a degree to which $A \Rightarrow B$ holds true in data. If T is crisp (which is particularly interesting) we write $A \Rightarrow B \in T$ if $T(A \Rightarrow B) = 1$ and $A \Rightarrow B \notin T$ if $T(A \Rightarrow B) = 0$.

For a fuzzy set T of fuzzy attribute implications, the set $\mathrm{Mod}(T)$ of all *models* of T is defined by

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}.$$

That is, $M \in \mathrm{Mod}(T)$ means that for each attribute implication $A \Rightarrow B$, a degree to which $A \Rightarrow B$ holds in M is higher than or at least equal to a degree $T(A \Rightarrow B)$ prescribed by T. Particularly, for a crisp T, $\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T : ||A \Rightarrow B||_M = 1\}$.

A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from a fuzzy set T of attribute implications is defined by

$$||A \Rightarrow B||_T = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M.$$

The following lemma was proved in [11].

Lemma 1. *For $A, B, M \in \mathbf{L}^Y$ and $c \in L$ we have*

$$c \leq ||A \Rightarrow B||_M \text{ iff } ||A \Rightarrow c \otimes B||_M = 1.$$

Lemma 1 has surprising consequences. It enables us to reduce the concept of a model of a fuzzy set of fuzzy attribute implications to the concept of a model of an ordinary set of fuzzy attribute implications, and to reduce the concept of semantic entailment from a fuzzy set of fuzzy attribute implications to the concept of semantic entailment from an ordinary set of fuzzy attribute implications:

Lemma 2. *Let T be a fuzzy set of fuzzy attribute implications and $A, B \in \mathbf{L}^Y$. Define an ordinary set $\mathrm{c}(T)$ of fuzzy attribute implications by*

$$\mathrm{c}(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in \mathbf{L}^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \emptyset\}. \tag{13}$$

Then we have

$$\mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{c}(T)), \tag{14}$$

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{c}(T)}. \tag{15}$$

Proof. (14) directly using Lemma 1. (15) is a consequence of (14).

Furthermore, Lemma 1 enables us to reduce the concept of a degree of entailment of a fuzzy attribute implication from a fuzzy set of fuzzy attribute implications to the concept of an entailment in degree 1 (full entailment) of a fuzzy attribute implication from a fuzzy set of fuzzy attribute implications:

Lemma 3. *For $A, B \in \mathbf{L}^Y$ and a fuzzy set T of fuzzy attribute implications we have*

$$||A \Rightarrow B||_T = \textstyle\bigvee\{c \in L \mid \; ||A \Rightarrow c \otimes B||_T = 1\}.$$

Proof. Using Lemma 1, we have

$$||A \Rightarrow B||_T = \bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M =$$
$$= \bigvee\{c \in L \mid c \leq ||A \Rightarrow B||_M \text{ for each } M \in \mathrm{Mod}(T)\} =$$
$$= \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_T = 1\}.$$

Therefore, we have:

Corollary 1. *For $A, B \in \mathbf{L}^Y$ and a fuzzy set T of fuzzy attribute implications we have*

$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_{\mathrm{c}(T)} = 1\},$$

with $\mathrm{c}(T)$ *defined by* (13).

Corollary 1 shows that the concept of a degree of entailment from a fuzzy set of fuzzy attribute implications can be reduced to entailment in degree 1 from a set of fuzzy attribute implications. We use this fact in the subsequent development.

An ordinary set T of fuzzy attribute implications is said to be *semantically closed* if $||A \Rightarrow B||_T = 1$ iff $A \Rightarrow B \in T$, i.e. if $T = \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$.

3.3 Completeness of fuzzy attribute logic

In this section, we introduce an axiomatic system for fuzzy attribute logic (FAL) and prove completeness theorems. First, we introduce deduction rules and a notion of a proof of a fuzzy attribute implication from an ordinary set T of fuzzy attribute implications. Second, we prove that a fuzzy attribute implication $A \Rightarrow B$ is provable from an ordinary set T of fuzzy attribute implications iff $A \Rightarrow B$ semantically follows from T in degree 1. Third, we introduce a concept of a degree $|A \Rightarrow B|_T$ of provability of a fuzzy attribute implication $A \Rightarrow B$ from a fuzzy set T of fuzzy attribute implications and show that $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$.

Axioms of FAL and some derived rules

Our axiomatic system consists of the following *deduction rules*.

(Ax) infer $A \cup B \Rightarrow A$,
(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,
(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Rules (Ax)–(Mul) are to be understood as usual deduction rules: having fuzzy attribute implications which are of the form of fuzzy attribute implication in the input part (the part preceding "infer") of a rule, a rule allows us to infer (in one step) the corresponding fuzzy attribute implication in the output part (the part following "infer") of a rule. (Ax) is a nullary rule (axiom) which says that each $A \cup B \Rightarrow A$ ($A, B \in \mathbf{L}^Y$) is inferred in one step.

Remark 3. (1) Rules (Ax) and (Cut) are inspired by [14]. The only difference from [14] is that A, B, C, D are fuzzy sets in (Ax) and (Cut) while in [14], A, B, C, D are ordinary sets.

(2) Rule (Mul) is a new rule in our fuzzy setting.

(3) If * is globalization, (Mul) can be omitted. Indeed, for $c = 1$, we have $c^* = 1$ and (Mul) becomes "from $A \Rightarrow B$ infer $A \Rightarrow B$" which is a trivial rule; for $c < 1$, we have $c^* = 0$ and (Mul) becomes "from $A \Rightarrow B$ infer $\emptyset \Rightarrow \emptyset$" which can be omitted since $\emptyset \Rightarrow \emptyset$ can be inferred by (Ax).

A fuzzy attribute implication $A \Rightarrow B$ is called *provable* from a set T of fuzzy attribute implications using a set $\mathcal{R}$ of deduction rules, written $T \vdash_{\mathcal{R}} A \Rightarrow B$, if there is a sequence $\varphi_1, \ldots, \varphi_n$ of fuzzy attribute implications such that φ_n is $A \Rightarrow B$ and for each φ_i we either have $\varphi_i \in T$ or φ_i is inferred (in one step) from some of the preceding formulas (i.e., $\varphi_1, \ldots, \varphi_{i-1}$) using some deduction rule from $\mathcal{R}$. If $\mathcal{R}$ consists of (Ax)–(Mul), we say just "provable ..." instead of "provable ... using $\mathcal{R}$" and write just $T \vdash A \Rightarrow B$ instead of $T \vdash_{\mathcal{R}} A \Rightarrow B$.

A deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" (φ_i, φ are fuzzy attribute implications) is said to be derivable from a set $\mathcal{R}$ of deduction rules if $\{\varphi_1, \ldots, \varphi_n\} \vdash_{\mathcal{R}} \varphi$. Again, if $\mathcal{R}$ consists of (Ax)–(Mul), we omit $\mathcal{R}$.

Lemma 4. *The following deduction rules are derivable from* (Ax) *and* (Cut):

(Ref) *infer* $A \Rightarrow A$,
(Wea) *from* $A \Rightarrow B$ *infer* $A \cup C \Rightarrow B$,
(Add) *from* $A \Rightarrow B$ *and* $A \Rightarrow C$ *infer* $A \Rightarrow B \cup C$,
(Pro) *from* $A \Rightarrow B \cup C$ *infer* $A \Rightarrow B$,
(Tra) *from* $A \Rightarrow B$ *and* $B \Rightarrow C$ *infer* $A \Rightarrow C$,

for each $A, B, C, D \in \mathbf{L}^Y$. *Moreover, if "from* $\varphi_1, \ldots, \varphi_n$ *infer* φ*" is a rule derivable from the ordinary Armstrong axioms* (*see* [24]) *then replacing symbols of sets by symbols of fuzzy sets, the rule is derivable from* (Ax) *and* (Cut).

Proof. First, we prove the second claim. The proof is almost trivial. Namely, it follows from [14] that each deduction rule derivable from the ordinary Armstrong axioms is derivable from (Ax$_c$) and (Cut$_c$) where (Ax$_c$) and (Cut$_c$) result from (Ax) and (Cut) by replacing fuzzy sets by ordinary sets. Now, observe that replacing ordinary sets with fuzzy sets in any proof from (Ax$_c$) and (Cut$_c$), we get a proof from (Ax) and (Cut).

The first claim is a consequence of the second one. Namely, each of the rules (Ref)–(Tra) is derivable from the original Armstrong axioms.

Completeness

We are now going to prove completeness of (Ax)–(Mul). Due to a lack of space we omit some technical details and restrict ourselves to the case of a finite $\mathbf{L}$.

A deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" is said to be *sound* if for each $M \in \text{Mod}(\{\varphi_1, \ldots, \varphi_n\})$ we have $M \in \text{Mod}(\{\varphi\})$, i.e. each model of all of $\varphi_1, \ldots, \varphi_n$ is also a model of φ.

Lemma 5. *Each of the deduction rules* (Ax)–(Mul) *is sound.*

Proof. For illustration, we check (Mul). Let $M \in \mathrm{Mod}(\{A \Rightarrow B\})$. We have to show that $M \in \mathrm{Mod}(\{c^* \otimes A \Rightarrow c^* \otimes B\})$.

First, $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff $||A \Rightarrow B||_M = 1$ iff $S(A, M)^* \leq S(B, M)$ iff

$$\text{for each } y \in Y: \ B(y) \otimes S(A, M)^* \leq M(y). \qquad (16)$$

Second, $M \in \mathrm{Mod}(\{c^* \otimes A \Rightarrow c^* \otimes B\})$ iff $||c^* \otimes A \Rightarrow c^* \otimes B||_M = 1$ iff $S(c^* \otimes A, M)^* \leq S(c^* \otimes B, M)$ iff for each $y \in Y$ we have

$$c^* \otimes B(y) \otimes S(c^* \otimes A, M)^* \leq M(y)$$

which is true. Indeed, using (16) we have

$$\begin{aligned}
&c^* \otimes B(y) \otimes S(c^* \otimes A, M)^* = \\
&= B(y) \otimes c^* \otimes \left(\textstyle\bigwedge_{y \in Y}((c^* \otimes A(y)) \to M(y))\right)^* = \\
&= B(y) \otimes c^* \otimes \left(\textstyle\bigwedge_{y \in Y}(c^* \to (A(y) \to M(y)))\right)^* = \\
&= B(y) \otimes c^* \otimes \left(c^* \to \textstyle\bigwedge_{y \in Y}(A(y) \to M(y))\right)^* = \\
&= B(y) \otimes c^* \otimes \left(c^* \to S(A, M)\right)^* \leq \\
&\leq B(y) \otimes c^* \otimes \left(c^{**} \to S(A, M)^*\right) = \\
&= B(y) \otimes c^* \otimes \left(c^* \to S(A, M)^*\right) \leq \\
&\leq B(y) \otimes S(A, M)^* \leq M(y).
\end{aligned}$$

We proved that (Mul) is sound.

Soundness of (Ax) and (Cut) can be proved analogously.

A set T of fuzzy attribute implications is said to be *syntactically closed* if $T \vdash A \Rightarrow B$ iff $A \Rightarrow B \in T$, i.e. if $T = \{A \Rightarrow B \,|\, T \vdash A \Rightarrow B\}$. The following lemma is almost immediate.

Lemma 6. *A set T of fuzzy attribute implications is syntactically closed iff we have:*

(Ax)-closure $A \cup B \Rightarrow A \in T$,
(Cut)-closure *if* $A \Rightarrow B \in T$ *and* $B \cup C \Rightarrow D \in T$ *then* $A \cup C \Rightarrow D \in T$,
(Mul)-closure *if* $A \Rightarrow B \in T$ *then* $c^* \otimes A \Rightarrow c^* \otimes B \in T$

for each $A, B, C, D \in \mathbf{L}^Y$, *and* $c \in L$.

Lemma 7. *Let T be a set of fuzzy attribute implications. If T is semantically closed then T is syntactically closed.*

Proof. By Lemma 6, we have to show that for each deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ", i.e. one of (Ax)–(Mul), we have that if $\varphi_1, \ldots \varphi_n \in T$ then $\varphi \in T$. Let thus $\varphi_1, \ldots \varphi_n \in T$. Since $\{\varphi_1, \ldots \varphi_n\} \subseteq T$, for any model $M \in \mathrm{Mod}(T)$ we have $M \in \mathrm{Mod}(\{\varphi_1, \ldots \varphi_n\})$, i.e. $M \in \mathrm{Mod}(\{\varphi_i\})$ for each $i = 1, \ldots, n$. Since each of the rules (Ax)–(Mul) is sound, we conclude $M \in \mathrm{Mod}(\{\varphi\})$. Since M is an arbitrary model of T, this shows that φ is true in each model of T. Since T is semantically closed, we get $\varphi \in T$.

Lemma 8. *Let T be a set of fuzzy attribute implications, let both Y and L be finite. If T is syntactically closed then T is semantically closed.*

Proof. Let T be syntactically closed. In order to show that T is semantically closed, it suffices to show $\{A \Rightarrow B \,|\, ||A \Rightarrow B||_T = 1\} \subseteq T$. We prove this by showing that if $A \Rightarrow B \notin T$ then $A \Rightarrow B \notin \{A \Rightarrow B \,|\, ||A \Rightarrow B||_T = 1\}$. Recall that since T is syntactically closed, T is closed under all of the rules (Ref)–(Tra) of Lemma 4.

Let thus $A \Rightarrow B \notin T$. To see $A \Rightarrow B \notin \{A \Rightarrow B \,|\, ||A \Rightarrow B||_T = 1\}$, we show that there is $M \in \mathrm{Mod}(T)$ which is not a model of $A \Rightarrow B$. For this purpose, consider $M = A^+$ where A^+ is the largest one such that $A \Rightarrow A^+ \in T$. A^+ exists. Namely, $S = \{C \,|\, A \Rightarrow C \in T\}$ is non-empty since $A \Rightarrow A \in T$ by (Ref), S is finite by finiteness of Y and L, and for $A \Rightarrow C_1, \ldots, A \Rightarrow C_n \in T$, we have $A \Rightarrow \bigcup_{i=1}^{n} C_i \in T$ by a repeated use of (Add).

We now need to check that (a) $||A \Rightarrow B||_{A^+} \neq 1$ (i.e., A^+ is not a model of $A \Rightarrow B$) and (b) for each $C \Rightarrow D \in T$ we have $||C \Rightarrow D||_{A^+} = 1$ (i.e., A^+ is a model of T).

(a): We need to show $||A \Rightarrow B||_{A^+} \neq 1$. By contradiction, suppose $||A \Rightarrow B||_{A^+} = 1$. Using $A \subseteq A^+$ we then get $1 = ||A \Rightarrow B||_{A^+} = S(A, A^+)^* \to S(B, A^+) = 1 \to S(B, A^+) = S(B, A^+)$, i.e. $B \subseteq A^+$. Since $A \Rightarrow A^+ \in T$, (Pro) would give $A \Rightarrow B \in T$, a contradiction.

(b): Let $C \Rightarrow D \in T$. We need to show $||C \Rightarrow D||_{A^+} = 1$, i.e. $S(C, A^+)^* \to S(D, A^+) = 1$ which is equivalent to $S(C, A^+)^* \otimes D \subseteq A^+$. To see this, it is sufficient to show that $A \Rightarrow S(C, A^+)^* \otimes D \in T$ (namely, A^+ is the largest one for which $A \Rightarrow A^+ \in T$). Note that we have (b1) $A \Rightarrow A^+ \in T$, (b2) $A^+ \Rightarrow S(C, A^+)^* \otimes C \in T$, and (b3) $S(C, A^+)^* \otimes C \Rightarrow S(C, A^+)^* \otimes D \in T$. Indeed, $A \Rightarrow A^+ \in T$ by definition of A^+; $A^+ \Rightarrow S(C, A^+)^* \otimes C \in T$ since as $S(C, A^+)^* \otimes C \subseteq A^+$, $A^+ \Rightarrow S(C, A^+)^* \otimes C$ in an instance of (Ax); and $S(C, A^+)^* \otimes C \Rightarrow S(C, A^+)^* \otimes D \in T$ by (Mul) applied to $C \Rightarrow D \in T$. Now, $A \Rightarrow S(C, A^+)^* \otimes D \in T$ follows by (Tra) applied twice to (b1), (b2), and (b3).

Corollary 2. *Let T be a set of fuzzy attribute implications. T is syntactically closed iff T is semantically closed.*

Theorem 1 (completeness). *Let $\mathbf{L}$ and Y be finite. Let T be a set of fuzzy attribute implications. Then*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1.$$

Proof. Sketch: Denote by $syn(T)$ the least syntactically closed set of fuzzy attribute implications which contains T. It can be shown that $syn(T) = \{A \Rightarrow B \,|\, T \vdash A \Rightarrow B\}$. Furthermore, denote by $sem(T)$ the least semantically closed set of fuzzy attribute implications which contains T. It can be shown that $sem(T) = \{A \Rightarrow B \,|\, ||A \Rightarrow B||_T = 1\}$. To prove the claim, we need to show $syn(T) = sem(T)$. As $syn(T)$ is syntactically closed, it is also semantically closed by Corollary 2 which means $sem(syn(T)) \subseteq syn(T)$. Therefore, by $T \subseteq syn(T)$ we get

$$sem(T) \subseteq sem(syn(T)) \subseteq syn(T).$$

In a similar manner we get $syn(T) \subseteq sem(T)$, showing $syn(T) = sem(T)$. The proof is complete.

Graded completeness

In this section, we define a notion of a degree $|A \Rightarrow B|_T$ of provability of a fuzzy attribute implication from a fuzzy set T of attribute implications. Then, we show that $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$ which can be understood as a graded completeness (completeness in degrees). Note that graded completeness was introduced by Pavelka [25], see also [16, 20] for further information.

For a fuzzy set T of fuzzy attribute implications and for $A \Rightarrow B$ we define a *degree* $|A \Rightarrow B|_T \in L$ *to which* $A \Rightarrow B$ *is provable from* T by

$$|A \Rightarrow B|_T = \bigvee\{c \in L \,|\, \mathrm{c}(T) \vdash A \Rightarrow c \otimes B\}, \tag{17}$$

where $\mathrm{c}(T)$ is defined by (13).

Theorem 2 (graded completeness). *Let* $\mathbf{L}$ *and* Y *be finite. Then for every fuzzy set* T *of fuzzy attribute implications and* $A \Rightarrow B$ *we have* $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$.

Proof. Consequence of Corollary 1 and Theorem 1.

3.4 Alternative axiomatizations and further derived rules

Our axioms (Ax), (Cut), and (Mul) have the following form: (Ax) and (Cut) result from an ordinary complete system by replacing sets by fuzzy sets and (Mul) is new axiom for fuzzy setting.

Note that for $\mathbf{L} = \mathbf{2}$ (two-element Boolean algebra, ordinary case), (Ax)–(Mul) form a complete system but (Mul) can be omitted since it either becomes "from $A \Rightarrow B$ infer $A \Rightarrow B$" (for $c = 1$), or "from $A \Rightarrow B$ infer $\emptyset \Rightarrow \emptyset$" (for $c = 0$), cf. Remark 3 (3).

As it can be expected, adding (Mul) to any system of deduction rules, which results from a complete system by replacing sets by fuzzy sets, yields a complete system:

Theorem 3. *Let* $\mathcal{R}_{\mathrm{Arm}}$ *be a system of deduction rules obtained from a complete system of ordinary deduction rules, i.e. equivalent to Armstrong axioms, by replacing symbols of sets by symbols of fuzzy sets. Then* (Ax)–(Mul) *are equivalent to rules consisting of those from* $\mathcal{R}_{\mathrm{Arm}}$ *plus* (Mul).

Proof. Follows from the fact that rules from $\mathcal{R}_{\mathrm{Arm}}$ are equivalent to (Ax) and (Cut).

Therefore, as in case of (Ax)–(Mul), we can have a system consisting of "ordinary deduction rules" and "fuzzy deduction rules". However, there are also complete systems with "mixed deduction rules". Consider the following rules (these rules were shown to be complete in [11]).

(Ax') infer $A \Rightarrow S(B, A) \otimes B$,
(Wea') from $A \Rightarrow B$ infer $A \cup C \Rightarrow B$,
(Cut') from $A \Rightarrow e \otimes B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow e^* \otimes D$

for each $A, B, C, D \in \mathbf{L}^Y$, and $e \in L$.

Theorem 4. (Ax)–(Mul) *are equivalent to* (Ax')–(Cut').

Proof. "$\Rightarrow$": (Ax') is an instance of (Ax) because $S(B, A) \otimes B \subseteq A$. (Wea') is equivalent to (Wea). (Cut'): let $\vdash A \Rightarrow e \otimes B$ and $\vdash B \cup C \Rightarrow D$. Then
$\vdash e^* \otimes (B \cup C) \Rightarrow e^* \otimes D$ by (Mul),
$\vdash (e \otimes B) \cup C \Rightarrow e^* \otimes D$ by (Wea),
$\vdash A \cup C \Rightarrow e^* \otimes D$ by (Cut). Hence, (Ax')–(Cut') can be derived from (Ax)–(Mul).

"$\Leftarrow$": (Ax) is of the form $A \Rightarrow B$, where $B \subseteq A$. Clearly, this is an instance of (Ax') for $S(B, A) = 1$. (Cut) is an instance of (Cut') for $e = 1$. (Mul): since $c^* \otimes A \Rightarrow c^* \otimes A$ is an instance of (Ax'), we get $c^* \otimes A \Rightarrow c^{**} \otimes B$ by (Cut') applied on $c^* \otimes A \Rightarrow c^* \otimes A$ and $A \Rightarrow B$; (5) gives that $c^* \otimes A \Rightarrow c^{**} \otimes B$ equals $c^* \otimes A \Rightarrow c^* \otimes B$ which is the desired formula. We showed that (Ax)–(Mul) can be derived from (Ax')–(Cut').

Another complete system of rules consists of (Ref), (Cut), and a rule

(S) from $A \Rightarrow B$ infer $C \Rightarrow S(A, C)^* \otimes B$,

where $A, B, C \in \mathbf{L}^Y$:

Theorem 5. (Ax)–(Mul) *are equivalent to* (Ref)*,* (Cut)*, and* (S)*.*

Proof. "$\Rightarrow$": Instead of directly showing the derivations of (Ref), (Cut), and (S) from (Ax)–(Mul), the fact that (Ref), (Cut), and (S) are derivable from (Ax)–(Mul) follows from completeness of (Ax)–(Mul) by observing that (Ref), (Cut), and (S) are sound (we omit the the proof soundness). Namely, soundness of a rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" implies

$$||\varphi||_{\{\varphi_1, \ldots, \varphi_n\}} = 1$$

from which we get by completeness of (Ax)–(Mul) that $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$, which means that a rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" is derivable from (Ax)–(Mul).

"$\Leftarrow$": By (Ref), we get $B \Rightarrow B$. Now, apply (S) to $B \Rightarrow B$ and $C = A \cup B$. Then $S(A, C)^* = S(B, A \cup B)^* = 1$ and thus (S) yields $A \cup B \Rightarrow B$ which shows that (Ax) is derivable. To see that (Mul) is derivable: From $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow S(A, c^* \otimes A)^* \otimes B$ by (S). Now, use (Ax) to infer $S(A, c^* \otimes A)^* \otimes B \cup c^* \otimes B \Rightarrow c^* \otimes B$ and observe that since $c^* \otimes B \subseteq S(A, c^* \otimes A)^* \otimes B$, we in fact inferred $S(A, c^* \otimes A)^* \otimes B \Rightarrow c^* \otimes B$. Since (Tra) is an instance of (Cut), we can apply it to see that $c^* \otimes A \Rightarrow c^* \otimes B$ can be inferred from $A \Rightarrow B$.

For $\mathbf{L}$-sets $A, B \in \mathbf{L}^Y$ define a degree $A \approx B$ of equality of A and B by

$$A \approx B = \bigwedge_{y \in Y} A(y) \approx B(y).$$

Note that $A \approx B$ is a truth degree of "for each $y \in Y$: y belongs to A iff y belongs to B" and that $\approx$ is a fuzzy equivalence relation [3]. Moreover, $A \approx B$ can be seen as a degree of similarity of A and B. It might be interesting to see what rules are derivable from (Ax)–(Mul) which say that from $A \Rightarrow B$ one can infer $C \Rightarrow D$ (or some modification of it) such that C is similar to A (or almost a subset or a superset of A) and D is similar to B (or almost a subset or a superset of B). The following theorem shows two examples.

Theorem 6. *The following deduction rules are derivable from* (Ax)–(Mul):

(Sub) *from* $A \Rightarrow B$ *infer* $C \Rightarrow D \otimes S(A,C)^* \otimes S(D,B)$,

(Sim) *from* $A \Rightarrow B$ *infer* $C \Rightarrow D \otimes (A \approx C)^* \otimes (D \approx B)$,

for each $A, B, C, D \in \mathbf{L}^Y$.

Proof. The assertion can be shown using completeness of (Ax)–(Mul) by observing that (Sub) and (Sim) are sound, cf. proof of Theorem 5.

Remark 4. (1) It can be shown that if $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ (i.e., $A \Rightarrow B$ is fully true in a data table $\langle X, Y, I\rangle$) then $||C \Rightarrow D||_{\langle X,Y,I\rangle} \geq (A \approx C)^* \otimes (D \approx B)$. This can be interpreted in the following way. If we would infer $C \Rightarrow D$ from $A \Rightarrow B$, our derivation would be "sound in degree at least $(A \approx C)^* \otimes (D \approx B)$". Theorem 6 shows that, if we infer $C \Rightarrow D \otimes (A \approx C)^* \otimes (D \approx B)$ instead, our derivation is sound.

(2) Note that (S) is an instance of (Sub) for $B = D$.

3.5 Concluding remarks

We showed a complete set of derivation rules for fuzzy attribute logic consisting of "ordinary rules" and "fuzzy rules". We proved both (the usual) completeness and graded completeness and showed some equivalent systems of derivation rules. Note that in [6, 8, 10] we described how to obtain a complete and non-redundant basis of the set of all fuzzy attribute implications which are fully true in a given data table with fuzzy attributes provided * is globalization and several other results concerning fuzzy attribute implications. Our future research will focus on:

- computation of non-redundant bases of fuzzy attribute implications for other hedges than globalization,
- relationships between fuzzy attribute logic and (ordinary) attribute logic (with the aim to study usual logical relationships like embedding etc.),
- relationships of fuzzy attribute logic and functional dependencies of database relations over domains with fuzzy similarity relations (first draft is [12]).

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