Galois Connections with Truth Stressers: Foundations for Formal Concept Analysis of Object-Attribute Data with Fuzzy Attributes

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1 Prologue: (Fuzzy) Galois Connections and Their Applications, and the Need for Further Development

Galois connections appear in several areas of mathematics and computer science, and their applications. A Galois connection between sets X and Y is a pair $\langle {}^\uparrow, {}^\downarrow \rangle$ of mappings ${}^\uparrow$ assigning subcollections of Y to subcollections of X, and ${}^\downarrow$ assigning subcollections of X to subcollections of Y. By definition, Galois connections have to satisfy certain conditions. Galois connections can be interpreted in the following manner: For subcollections A and B of X and Y, respectively, $A^\uparrow$ is the collection of all elements of Y which are in a certain relationship to all elements from A, and $B^\downarrow$ is the collection of all elements of X which are in the relationship to all elements in B. From the very many examples of Galois connections in mathematics, let us recall the following. Let X be the set of all logical formulas of a given language, Y be the set of all structures (interpretations) of the same language. For $A \subseteq X$ and $B \subseteq Y$, let $A^\uparrow$ consist of all structures in which each formula from A is true, let $B^\downarrow$ denote the set of all formulas which are true in each structure from B. Then, ${}^\uparrow$ and ${}^\downarrow$ is a Galois connection.

As an example of applications of Galois connections, consider the following example which is the main source of inspiration for the present paper. Let X and Y denote a set of objects and attributes, respectively, Let I denote the relationship "to have" between objects and attributes. Then X, Y, and I can be seen as representing an object-attribute data table (for instance, organisms as objects, and their properties as attributes). If, for subcollections A of X and B of Y, $A^\uparrow$ denotes the collection of all attributes shared by all objects from A, and $B^\downarrow$ denotes the collection of all objects sharing all attributes from B, then ${}^\uparrow$ and ${}^\downarrow$ form a Galois connection. These connections form the core of so-called formal concept analysis (FCA) of object-attribute data, see [18]. Fixed

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points of these connections, i.e. pairs $\langle A, B\rangle$ for which $A^{\uparrow} = B$ and $B^{\downarrow} = A$, are called formal concepts and represent interesting clusters found in the data table (formal concepts in the above-mentioned table with organisms and their properties may be mammals, warm-blooded organisms, etc.). Formal concepts can be partially ordered by subconsept-superconcept hierarchy (a concept can be more general or more particular than a given concept). For instance, the concept "mammal" is more general than "dog". The hierarchically ordered set of all formal concepts, so-called concept lattice, provide us with a derived conceptual information hidden in the data. Formal concept analysis can be thought of as directly formalizing the ideas on what are concepts as developed by so-called Port-Royal logic [1]. FCA has found applications in several areas (software engineering, psychology, text classification, reengineering).

Galois connections have been explicitly introduced in [26]. After some suggestions to use Galois connections for data analysis by Birkhoff, the first systematic paper on data analysis using Galois connections is [3], see also [4]. Probably the most influential paper in FCA is Wille's [29] which started an intensive research on FCA. In basic setting, FCA deals with bivalent attributes, i.e. each object either has (degree 1) or does not have (degree 0) a given attribute. In order to deal with fuzzy (graded) attributes, FCA has been generalized in several papers, see e.g. [5, 16, 27]. A fuzzy attribute can apply to an object to a degree in between 0 and 1 (e.g. 0.3), i.e not only 0 or 1 as in case of bivalent attributes. Galois connections generalized from the point of view of fuzzy approach so that they correspond to FCA of data with fuzzy attributes have been introduced in [6]; fuzzy concept lattices, i.e. fixed points of fuzzy Galois connections, have been studied in [7, 13].

The main motivation of the present paper stems from [14] where the authors showed a way to reduce the number of formal concepts in FCA with fuzzy attributes by considering only so-called crisply generated formal concepts. Crisply generated concepts can be considered as natural concepts with clear interpretation. Moreover, as shown in [14], they can be efficiently generated (without the need to generate all formal concepts and to test whether a particular concept is crisply generated). Now, the question is whether crisply generated formal concepts are fixed points of structures analogous to Galois connections. The present paper gives a positive answer to this question, and elaborates more on the presented topic and some related problems.

2 Preliminaries

We pick complete residuated lattices as the structures of truth values. Complete residuated lattices, being introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [19]. Various logical calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough information about the role of residuated lattices in fuzzy logic can be obtained in [20, 21, 25]. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is a commutative and associative binary operation on L satisfying $a \otimes 1 = a$), and $\otimes$, $\rightarrow$ form an adjoint pair, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$. In the following, $\mathbf{L}$ denotes an arbitrary complete residuated lattice (with L being the universe set of $\mathbf{L}$). All properties of complete residuated lattices used in the sequel are well-known and can be found e.g. in [11]. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [21, 23]).

Of particular interest are complete residuated lattices defined on the real unit interval $[0, 1]$ or on some subchain of $[0, 1]$. It can be shown (see e.g. [11]) that $\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $\rightarrow$ is defined by $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$. A t-norm is a binary operation on $[0, 1]$ which is associative, commutative, monotone, and has 1 as its neutral element, and hence, captures the basic properties of conjunction. A t-norm is called left-continuous if, as a real function, it is left-continuous in both arguments. Most commonly used are continuous t-norms, the basic three of which are Łukasiewicz t-norm (given by $a \otimes b = \max(a + b - 1, 0)$ with the corresponding residuum $a \rightarrow b = \min(1 - a + b, 1)$), minimum (also called Gödel) t-norm ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product t-norm ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). It can be shown (see e.g. [24]) that each continuous t-norm is composed out of the three above-mentioned t-norms by a simple construction (ordinal sum). Any finite subchain of $[0, 1]$ containing both 0 and 1, equipped with restrictions of the minimum t-norm and its residuum is a complete residuated lattice. Furthermore, the same holds true for any equidistant finite chain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ equipped with restrictions of Łukasiewicz operations. The only residuated lattice on the two-element chain $\{0, 1\}$ (with $0 < 1$) has the classical conjunction operation as $\otimes$ and classical implication operation as $\rightarrow$. That is, the two-element residuated lattice is the two-element Boolean algebra of classical logic.

A fuzzy set with truth degrees from a complete residuated lattice $\mathbf{L}$ (also simply an $\mathbf{L}$-set) in a universe set X is any mapping $A \colon X \rightarrow L$, $A(x) \in L$ being interpreted as the truth value of "x belongs to A".

Analogously, an n-ary $\mathbf{L}$-relation on a universe set X is an $\mathbf{L}$-set in the universe set X^n, e.g. a binary relation R on X is a mapping $R \colon X \times X \rightarrow L$. A singleton is a fuzzy set $\{{}^{a}/x\}$ for which $\{{}^{a}/x\}(x) = a$ and $\{{}^{a}/x\}(y) = 0$ for $y \neq x$. A fuzzy set A is called normal if $A(x) = 1$ for some $x \in X$. For $a \in L$, the a-cut of a fuzzy set $A \in L^X$ is the ordinary subset ${}^{a}A = \{x \in X \mid A(x) \geq a\}$ of X. For $\mathbf{L}$-sets A and B in X we define $(A \approx B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$ (degree of equality of A and B) and $S(A, B) = \bigwedge_{x \in X}(A(x) \rightarrow B(x))$ (degree of subsethood of A in B). Note that $\leftrightarrow$ is defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$.

Clearly, $(A \approx B) = S(A,B) \wedge S(B,A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A,B) = 1$, i.e. for each $x \in X$, $A(x) \leq B(x)$. $A \subset B$ means $A \subseteq B$ and $A \neq B$. The set of all **L**-sets in X will be denoted by L^X. Note that the operations of **L** induce the corresponding operations on L^X. For example, we have intersection $\bigcap$ on L^X induced by the infimum $\bigwedge$ of **L** by $(\bigcap_{i\in I} A_i)(x) = \bigwedge_{i\in I} A_i(x)$, etc. A fuzzy set A in X is called crisp if $A(x) = 0$ or $A(x) = 1$ for each $x \in X$. In this case, we write also $A \subseteq X$ since A may be obviously identified with an ordinary subset of X.

3 Galois Connections with Truth Stressers

3.1 Coming to Galois Connections with Truth Stressers

Fuzzy Galois connections and concept lattices

Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \to L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has attribute y (L is a support set of some complete residuated lattice **L**). The triplet $\langle X, Y, I\rangle$ is called a formal fuzzy context.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} (A(x) \to I(x,y)) \tag{1}$$

and

$$B^{\downarrow}(x) = \bigwedge_{y\in Y} (B(y) \to I(x,y)). \tag{2}$$

Using basic rules of predicate fuzzy logic [11], one can easily see that $A^{\uparrow}(y)$ is the truth degree of the fact "y is shared by all objects from A" and $B^{\downarrow}(x)$ is the truth degree of the fact "x has all attributes from B". Putting

$$\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$

$\mathcal{B}(X,Y,I)$ is the set of all pairs $\langle A,B\rangle$ such that (a) A is the collection of all objects that have all the attributes of (the intent) B and (b) B is the collection of all attributes that are shared by all the objects of (the extent) A. Elements of $\mathcal{B}(X,Y,I)$ are called formal concepts of $\langle X,Y,I\rangle$; $\mathcal{B}(X,Y,I)$ is called the concept lattice given by $\langle X,Y,I\rangle$. Both the extent A and the intent B of a formal concept $\langle A,B\rangle$ are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to various intermediate degrees, not only 0 and 1.

Putting

$$\langle A_1, B_1\rangle \leq \langle A_1, B_1\rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_1 \supseteq B_2) \tag{3}$$

for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$, $\leq$ models the subconcept-superconcept hierarchy in $\mathcal{B}(X, Y, I)$. That is, being more general means to apply to a larger collection of objects and to cover a smaller collection of attributes. Characterization of $\mathcal{B}(X, Y, I)$ is presented in [13], see also [7, 27].

Given $\langle X, Y, I\rangle$, the pair $\langle^{\uparrow}, ^{\downarrow}\rangle$ induced by (1) and (2) satisfies the following natual properties [6]:

$$S(A_1, A_2) \leq S(A_2^{\uparrow}, A_1^{\uparrow}) \tag{4}$$

$$S(B_1, B_2) \leq S(B_2^{\downarrow}, B_1^{\downarrow}) \tag{5}$$

$$A \subseteq A^{\uparrow\downarrow} \tag{6}$$

$$B \subseteq B^{\downarrow\uparrow}, \tag{7}$$

for each $A, A_1, A_2 \in L^X$ and $B, B_1, B_2 \in L^Y$. A pair $\langle^{\uparrow}, ^{\downarrow}\rangle$ satisfying (4)–(7) is called a fuzzy Galois connection. It was proved in [6] that each fuzzy Galois connection is induced by some $\langle X, Y, I\rangle$ by (1) and (2). Note that fuzzy Galois connection obey several further properties which are often used, e.g. $A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$.

Crisply generated formal concepts

An important problem in FCA is a possible large number of formal concepts in $\mathcal{B}(X, Y, I)$. A way to cope with this problem in case of data with fuzzy attributes was proposed in [14]. The following are the basics. A formal concept $\langle A, B\rangle$ consists of a fuzzy set A and a fuzzy set B such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$ which directly captures the verbal definition of a formal concept inspired by Port-Royal logic. However, this definition might actually allow for formal fuzzy concepts which seem not natural. For example, there may exist a formal fuzzy concept $\langle A, B\rangle$ such that for any $x \in X$ and $y \in Y$ we have $A(x) = 1/2$ and $B(y) = 1/2$. A verbal description of such a concept is "a concept to which each attribute belongs to degree 1/2". In general, "a concept to which each attribute belongs to degree 1/2" might be difficult to interpret. This is because people expect concepts to be determined by "some attributes", i.e. by an ordinary set of attributes. This leads to the following definition.

Definition 1. *A formal fuzzy concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is called* crisply generated *if there is a crisp set $B_c \subseteq Y$ such that $A = B_c^{\downarrow}$ (and thus $B = B_c^{\downarrow\uparrow}$). We say that B_c crisply generates $\langle A, B\rangle$.*

By $\mathcal{B}_c(X, Y, I)$ we denote the collection of all crisply generated formal concepts in $\langle X, Y, I\rangle$, i.e.

$$\mathcal{B}_c(X, Y, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid \text{ there is } B_c \subseteq Y : A = B_c^{\downarrow}\}.$$

That is, $\mathcal{B}_c(X, Y, I) = \{\langle B_c^{\downarrow}, B_c^{\downarrow\uparrow}\rangle \mid B_c \subseteq Y\}$.

For further information on crisply generated fuzzy concepts, demonstration of the reduction of the number of formal concepts, and an algorithm for listing all crisply generated concepts we refer to [14].

To better understand the structure of crisply generated concepts and their further properties, the following question arises:

> Is there some "Galois-connection-like" structure behind crisply generated concepts which plays the role analogous to the role of Galois connections in FCA?

In the following, we elaborate the basic answer, some partial answers, and outline some open problems and directions.

3.2 Case One: Galois Connections Behind Crisply Generated Formal Concepts

Galois connections with truth stressers

First, we provide another view on crisply generated concepts which turns out to be suitable for our purposes. We will need the notion of a truth stresser. A truth stresser is a unary function * on the set L of truth degrees with the following interpretation: For a truth degree $a \in L$, the value $a^* \in L$ is the degree to which a can be considered as very true. Formally, a truth stresser on a structure $\mathbf{L}$ of truth degrees is a unary function which is required to satisfy some natural properties, e.g. $a^* \leq a$; $a \leq b$ implies $a^* \leq b^*$; $1^* = 1$; $a^* = a^{**}$. Functions with properties of truth stressers were used in [28]. In the context of fuzzy logic, truth stressers go back to [2] and were further elaborated in [21, 22]. For simplicity and because of our motivation by crisply generated concepts, we use only a particular type of a truth stresser in the present paper. Namely, we use what we call a Baaz operator [2] which is a function $^* : L \to L$ defined by

$$a^* = \begin{cases} 1 \text{ for } a = 1 \\ 0 \text{ for } a \neq 1. \end{cases} \tag{8}$$

Throuhgout the rest of the paper, * denotes the Baaz operator (8).

Consider now the mappings $^\triangle : L^X \to L^Y$ and $^\triangledown : L^Y \to L^X$ resulting from $\langle X, Y, I \rangle$ by

$$A^\triangle(y) = \bigwedge_{x \in X} (A(x) \to I(x, y)) \tag{9}$$

and

$$B^\triangledown(x) = \bigwedge_{y \in Y} (B(y)^* \to I(x, y)). \tag{10}$$

Remark 1. (1) Note that we have $A^{\triangle} = A^{\uparrow}$ and $B^{\triangledown} = (B^*)^{\downarrow}$ where $B^*(y) = (B(y))^*$, and $^{\uparrow}$ and $^{\downarrow}$ are defined by (1) and (2).

(2) With regard to the interpretation of a truth stresser *, $B^{\triangledown}(x)$ is the truth degree of "for each y: if it is very true that y belongs to B then x has (attribute) y". The particular meaning depends on the truth stresser *. For Baaz operator (8), this reads "for each y: if y fully belongs to B (i.e., belongs in degree 1) then x has y".

(3) Although we do not consider other truth stressers than (8) in this paper, let us note that another example of a truth stresser is the identity on L, i.e. $a^* = a$. For this choice we clearly have $A^{\triangle} = A^{\uparrow}$ and $B^{\triangledown} = B^{\downarrow}$. Therefore, fuzzy Galois connections result by a particular choice of a truth stresser.

The main points we are going to show in the rest of this section are, first, that crisply generated concepts are exactly fixed points of $\langle^{\triangle}, ^{\triangledown}\rangle$, and, second, that $^{\triangle}$ and $^{\triangledown}$ can be defined axiomatically.

Crisply generated concepts as fixed points of $\langle^{\triangle}, ^{\triangledown}\rangle$

For * defined by (8) denote by $\mathcal{B}(X, Y^*, I)$ the set of all fixed points of $\langle^{\triangle}, ^{\triangledown}\rangle$, i.e.

$$\mathcal{B}(X, Y^*, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\triangle} = B,\ B^{\triangledown} = A\}.$$

The following theorem shows that $\mathcal{B}(X, Y^*, I)$ are exactly the crisply generated concepts in $\langle X, Y, I\rangle$.

Theorem 1. *For a truth stresser* * *defined by (8),* $\mathcal{B}(X, Y^*, I) = \mathcal{B}_c(X, Y, I)$.

Proof. "$\subseteq$": If $\langle A, B\rangle \in \mathcal{B}(X, Y^*, I)$ then $A^{\triangle} = B$ and $B^{\triangledown} = A$, i.e. $A^{\uparrow} = B$ and $B^{*\downarrow} = A$. From (8) we get that B^* is crisp (i.e. $B^*(y)$ is 0 or 1 for each $y \in Y$). Therefore, $\langle A, B\rangle \in \mathcal{B}_c(X, Y, I)$, by definition.

"$\supseteq$": Let $\langle A, B\rangle \in \mathcal{B}_c(X, Y, I)$, i.e. $A^{\uparrow} = B$, $B^{\downarrow} = A$, and $A = D^{\downarrow}$ for some $D \subseteq Y$. We need to verify $A^{\triangle} = B$ and $B^{\triangledown} = A$, for which it clearly suffices to check $B^{\triangledown} = A$, i.e. $B^{*\downarrow} = A$. Since $A = D^{\downarrow}$ and $B = B^{\downarrow\uparrow}$, we need to check $D^{\downarrow} = D^{\downarrow\uparrow^{*\downarrow}}$. Now observe that we have $D^{\downarrow} = D^{*\downarrow} = D^{*\downarrow\uparrow^{*\downarrow}}$. Indeed, the first equality follows from the fact that D is crisp and thus $D^* = D$. For the second equality, $D^{*\downarrow} \subseteq D^{*\downarrow\uparrow^{*\downarrow}}$ follows from $F \subseteq F^{\uparrow^{*\downarrow}}$ for $F = D^{*\downarrow}$ (easy to verify), while $D^{*\downarrow} \supseteq D^{*\downarrow\uparrow^{*\downarrow}}$ follows from $D = D^*$, from $D^* \subseteq D^{*\downarrow\uparrow}$, and from the fact that if $E \subseteq F$ then $E^{*\downarrow} \supseteq F^{*\downarrow}$ (just put $E = D$ and $F = D^{*\downarrow\uparrow}$). We showed $\langle A, B\rangle \in \mathcal{B}(X, Y^*, I)$, finishing the proof.

Galois connections with truth stressers: $^{\triangle}$ *and* $^{\triangledown}$ *defined axiomatically*

We now turn to the investigation of the properties of $^{\triangle}$ and $^{\triangledown}$ with the aim to provide a simple axiomatic characterization.

Lemma 1. *Let* * *be defined by (8). Then* $^\triangle$ *and* $^\triangledown$ *defined by (9) and (10) satisfy*

$$S(A, B^\triangledown) = S(B^*, A^\triangle) \tag{11}$$

$$(\bigcup_{j\in J} A_j)^\triangle = \bigcap_{j\in J} A_j^\triangle \tag{12}$$

for every $A, A_j \in L^X$ *and* $B \in L^Y$.

Proof. We have

$$\begin{aligned} S(A, B^\triangledown) &= \bigwedge_{x\in X} A(x) \to (\bigwedge_{y\in Y} B^*(y) \to I(x,y)) = \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y} A(x) \to (B^*(y) \to I(x,y)) = \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X} B^*(y) \to (A(x) \to I(x,y)) = \\ &= \bigwedge_{y\in Y} B^*(y) \to (\bigwedge_{x\in X} A(x) \to I(x,y)) = \\ &= S(B^*, A^\triangle), \end{aligned}$$

proving (11). As $^\triangle = {}^\uparrow$, (12) is a consequence of properties of fuzzy Galois connections [6].

Definition 2. *A pair* $\langle ^\triangle, ^\triangledown \rangle$ *of mappings satisfying (11) and (12) is called a fuzzy Galois connection with truth stresser* *.

The following are some consequences of (11).

Lemma 2. *If for* * *defined by (8) mappings* $^\triangle : L^X \to L^Y$ *and* $^\triangledown : L^Y \to L^X$ *satisfy (11) then*

$$(\bigcup_{j\in J} B_j{}^*)^\triangledown = \bigcap_{j\in J} B_j^\triangledown \tag{13}$$

$$B^\triangledown = B^{*\triangledown} \tag{14}$$

$$\{\,{}^a/x\}^\triangle(y) = a \to \{\,{}^1/x\}^\triangle(y) \tag{15}$$

$$\{\,{}^a/y\}^\triangledown(x) = a \to \{\,{}^1/y\}^\triangledown(x) \tag{16}$$

for any $B, B_j \in L^Y$, $x \in X$, $y \in Y$, $a \in L$.

Proof. We show (13) by showing that $S(A, (\bigcup_i B^*{}_i)^\triangledown) = 1$ iff $S(A, \bigcap_i B^{*\triangledown}_i) = 1$ for each $A \in L^X$. First note that using (11) we have

$$S(A, (\bigcup_i B^*{}_i)^\triangledown) = S((\bigcup_i B^*{}_i)^*, A^\triangle) = S((\bigcup_i B^*{}_i), A^\triangle).$$

As a results, we have $S(A, (\bigcup_i {B^*}_i)^\bigtriangledown) = 1$ iff $S((\bigcup_i {B^*}_i), A^\bigtriangleup) = 1$ iff for each i we have ${B^*}_i \subseteq A^\bigtriangleup$ iff for each i we have $S({B^*}_i, A^\bigtriangleup)$ iff for each i we have $S(A, B_i^\bigtriangledown)$ iff $S(A, \bigcap_i B_i^\bigtriangledown)$, showing (13).

(14) follows from (13) for $|J| = 1$.

(15) and (16) follow from $b^* \to \{\,{}^{a}\!/x\}^\bigtriangleup(y) = a \to \{\,{}^{b}\!/y\}^\bigtriangledown(x)$ and $\{\,{}^{1}\!/x\}^\bigtriangleup(y) = \{\,{}^{1}\!/y\}^\bigtriangledown(x)$ which we now verify. First,

$$b^* \to \{\,{}^{a}\!/x\}^\bigtriangleup(y) = S(\{\,{}^{b^*}\!/y\}, \{\,{}^{a}\!/x\}^\bigtriangleup) = S(\{\,{}^{b}\!/y\}^*, \{\,{}^{a}\!/x\}^\bigtriangleup) =$$
$$= S(\{\,{}^{a}\!/x\}, \{\,{}^{b}\!/y\}^\bigtriangledown) = a \to \{\,{}^{b}\!/y\}^\bigtriangledown(x).$$

Second, $\{\,{}^{1}\!/x\}^\bigtriangleup(y) = \{\,{}^{1}\!/y\}^\bigtriangledown(x)$ is a consequence of the first equality for $a = b = 1$.

Lemma 3. *Let* * *be defined by (8). Let* $\langle^\bigtriangleup, ^\bigtriangledown\rangle$ *be a fuzzy Galois connection with* *. *Then there is a fuzzy relation* $I \in L^{X\times Y}$ *such that* $\langle^\bigtriangleup, ^\bigtriangledown\rangle = \langle^{\bigtriangleup_I}, ^{\bigtriangledown_I}\rangle$ *where* $^{\bigtriangleup_I}$ *and* $^{\bigtriangledown_I}$ *are induced by* I *by (9) and (10).*

Proof. Let I be defined by $I(x,y) = \{\,{}^{1}\!/x\}^\bigtriangleup(y) = \{\,{}^{1}\!/y\}^\bigtriangledown(x)$. Then using (15)

$$A^\bigtriangleup(y) = (\bigcup_{x\in X}\{\,{}^{A(x)}\!/x\})^\bigtriangleup(y) =$$
$$= (\bigcap_{x\in X}\{\,{}^{A(x)}\!/x\}^\bigtriangleup)(y) = \bigwedge_{x\in X}\{\,{}^{A(x)}\!/x\}^\bigtriangleup(y) =$$
$$= \bigwedge_{x\in X} A(x) \to \{\,{}^{1}\!/x\}^\bigtriangleup(y) = \bigwedge_{x\in X} A(x) \to I(x,y) = A^{\bigtriangleup_I}(y).$$

Furthermore, using (13) and (14), and (16) we get

$$B^\bigtriangledown(x) = {B^*}^\bigtriangledown(x) = (\bigcup_{y\in Y}\{\,{}^{B^*(y)}\!/y\})^\bigtriangledown(x) =$$
$$= (\bigcup_{y\in Y}\{\,{}^{B(y)}\!/y\}^*)^\bigtriangledown(x) = (\bigcap_{y\in Y}\{\,{}^{B(y)}\!/y\}^\bigtriangledown)(x) = \bigwedge_{y\in Y}\{\,{}^{B(y)}\!/y\}^\bigtriangledown(x) =$$
$$= \bigwedge_{y\in Y} B^*(y) \to \{\,{}^{1}\!/y\}^\bigtriangledown(x) = \bigwedge_{y\in Y} B^*(y) \to I(x,y) = B^{\bigtriangledown_I}(x).$$

The following theorem shows that there is a one-to-one correspondence between fuzzy Galois connections with * defined by (8).

Theorem 2. *Let* $I \in L^{X\times Y}$ *be a fuzzy relation, let* $^{\bigtriangleup_I}$ *and* $^{\bigtriangledown_I}$ *be defined by (9) and (10). Let* $\langle^\bigtriangleup, ^\bigtriangledown\rangle$ *be a fuzzy Galois connection with* * *defined by (8). Then*

(1) $\langle^{\bigtriangleup_I}, ^{\bigtriangledown_I}\rangle$ *satisfy (11) and (12).*
(2) $I_{\langle^\bigtriangleup, ^\bigtriangledown\rangle}$ *defined as in the proof of Lemma 3 is a fuzzy relation and we have*

(3) $\langle{}^{\triangle},{}^{\triangledown}\rangle = \langle{}^{\triangle_{I_{\langle\triangle,\triangledown\rangle}}},{}^{\triangledown_{I_{\langle\triangle,\triangledown\rangle}}}\rangle$ *and* $I = I_{\langle\triangle_I,\triangledown_I\rangle}$.

Proof. Due to the previous results, it remains to check $I = I_{\langle\triangle_I,\triangledown_I\rangle}$. We have

$$\begin{aligned} I_{\langle\triangle_I,\triangledown_I\rangle}(x,y) &= \{\,{}^1/x\}^{\triangle_I}(y) = \\ &= \bigwedge_{z\in X}\{\,{}^1/x\}(z) \to I(z,y) = I(x,y). \end{aligned}$$

Let us consider conditions (4)–(7). These are the defining conditions for fuzzy Galois connections. However, for Galois connections with truth stresser (8), (5) and (7) are not satisfied, as shown by the following example.

Example 1. Take $X = \{x\}, Y = \{y\}$, $I(x,y) = 0.3$, $B_1(y) = 1$, $B_2(y) = 0.8$, take $L = [0,1]$ equipped with the Łukasiewicz structure, and * defined by (8).
(5): We have $0.8 = S(B_1,B_2) \not\leq S(B_2^{\triangledown},B_1^{\triangledown}) = 0.3$, a counterexample to (5).
(7): We have $B(y) = 0.8 \not\leq 0.3 = B^{\triangledown\triangle}(y)$, i.e. $\langle{}^{\triangle},{}^{\triangledown}\rangle$ does not satisfy (7).

The next lemma shows properties of Galois connections with * which are analogous to (4)–(7).

Lemma 4. *If a pair* $\langle{}^{\triangle},{}^{\triangledown}\rangle$ *is a fuzzy Galois connection with* * *defined by (8) then*

$$S(A_1,A_2) \leq S(A_2^{\triangle},A_1^{\triangle}) \tag{17}$$

$$S(B_1{}^*,B_2{}^*) \leq S(B_2^{\triangledown},B_1^{\triangledown}) \tag{18}$$

$$A \subseteq A^{\triangle\triangledown} \tag{19}$$

$$B^* \subseteq B^{\triangledown\triangle} \tag{20}$$

Proof. The assertion follows from the properties of $\langle{}^{\uparrow},{}^{\downarrow}\rangle$, Remark 1 (1), and from the fact that $A^{\uparrow} \subseteq A^{\triangle}$ (we omit details).

3.3 Case Two, Three, ..., and Others

Placement of the truth stresser

The particular case of fuzzy Galois connection with a truth stresser * may be considered just one out of several further possibilities. Namely, the placement of the truth stresser *, inspired by [14] is not the only possible. The aim of this section is to outline some possibilities with some results. However, due to the limited scope of the paper, this section is to be considered only a sketch of a more detailed study which is under preparation and is to be published.

From the epistemic point of view, various placements of truth stressers in formulas which define ${}^{\triangle}$ and ${}^{\triangledown}$ lead to various interpretations of concepts. For instance, the concepts determined by the placement introduced in Section 3.2 can be interpreted as "crisply generated by attributes". In much the same way we can define concepts (i) "crisply generated by objects", (ii) "crisply generated by object and atributes", etc. In general, it seems reasonable to consider $^{\triangle}$ and $^{\triangledown}$ defined by

$$A^{\triangle}(y) = \bigwedge_{x\in X} A^{*_1}(x) \rightarrow I^{*_2}(x,y),$$
$$B^{\triangledown}(x) = \bigwedge_{y\in Y} B^{*_3}(y) \rightarrow I^{*_4}(x,y),$$

where $^{*_1},\dots,^{*_4}$ are appropriate truth stressers. Taking ^{*_3} to be the Baaz operator (8) and taking the identity for $^{*_1},^{*_2},^{*_4}$, we obtain Galois connections defined studied in Section 3.2.

A problem that offers itself is to take systems of all formal concepts (fixed points) determined by various placements and study their relationships. The problem is especially interesting for applications of FCA, because some placements can lead to smaller (and yet natural) sets of formal concepts—this might be understood as a purely logical way to reduce the size of the resulting conceptual structure.

Note first that taking other truth stresser than the identity for ^{*_2} and ^{*_4}, we lose the possibility to obtain I back from suitable axiomatic properties of $^{\triangle}$ and $^{\triangledown}$. Suppose for simplicity $^{*_2} = {}^{*_4} = {}^{*}$. Then the same $^{\triangle}$ and $^{\triangledown}$ are clearly induced by any fuzzy relation J for which $I^* = J^*$. Therefore, if one wants to keep the one-to-one relationship between I's and $\langle^{\triangle},^{\triangledown}\rangle$, one has to restrict the attention to fuzzy relations for which $I^* \neq J^*$ for $I \neq J$ (a natural choice in case of idempotent * seems to be the set $\{I^* \mid I \in L^{X\times Y}\}$).

Crisply generated concepts, by attributes, objects, and both

In the following we present some preliminary results for the case when both ^{*_2} and ^{*_4} are identities, and ^{*_1} and ^{*_3} are either the identity or the Baaz operator (8). Given a fuzzy context $\langle X, Y, I\rangle$, we consider the following subsets of $L^X \times L^Y$:

$$\mathcal{B}(X, Y^*, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\, B^{*\downarrow} = A\},$$
$$\mathcal{B}(X^*, Y, I) = \{\langle A, B\rangle \mid A^{*\uparrow} = B,\, B^{\downarrow} = A\},$$
$$\mathcal{B}(X^*, Y^*, I) = \{\langle A, B\rangle \mid A^{*\uparrow} = B,\, B^{*\downarrow} = A\}.$$

Clearly, the above definition of $\mathcal{B}(X, Y^*, I)$ coincides with that one presented in Section 3.2. On the verbal level, we can call $\mathcal{B}(X, Y^*, I)$, $\mathcal{B}(X^*, Y, I)$, and $\mathcal{B}(X^*, Y^*, I)$ collections of all formal concepts crisply generated by *attributes, objects,* and *attributes and objects,* respectively.

Theorem 3. *For a truth stresser * defined by (8), we have*

(i) *if $\langle A, B\rangle \in \mathcal{B}(X, Y^*, I) \cap \mathcal{B}(X^*, Y, I)$ then $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$;*
(ii) *if $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$ then*
$\langle A, A^{\uparrow}\rangle \in \mathcal{B}(X, Y^*, I)$ and $\langle B^{\downarrow}, B\rangle \in \mathcal{B}(X^*, Y, I)$;

(iii) $|\mathcal{B}(X^*, Y^*, I)| \leq \min(|\mathcal{B}(X, Y^*, I)|, |\mathcal{B}(X^*, Y, I)|)$.

Proof. (i) follows directly by definition.

(ii): Assuming $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$, we have $A^{*\uparrow} = B$ and $B^{*\downarrow} = A$. We show that $\langle A, A^{\uparrow}\rangle \in \mathcal{B}(X, Y^*, I)$. It suffices to check $A^{\uparrow*\downarrow} = A$ since the rest follows easily. The monotony gives $A \subseteq A^{\uparrow\downarrow} \subseteq A^{\uparrow*\downarrow}$. Conversely, by the idempotency of * we have $A^{\uparrow*\downarrow} = B^{*\downarrow\uparrow*\downarrow} \subseteq B^{*\downarrow} = A$, showing that $\langle A, A^{\uparrow}\rangle \in \mathcal{B}(X, Y^*, I)$. Dually, one can prove $\langle B^{\downarrow}, B\rangle \in \mathcal{B}(X^*, Y, I)$.

(iii) is a consequence of (ii). □

Note that (iii) of Theorem 3 says that the collection of all concepts crisply generated by attributes and objects cannot be strictly greater than the collection of all concepts crisply generated by attributes (objects). If * were not idempotent this property would not hold in general.

Given a fuzzy context $\langle X, Y, I\rangle$ and * defined by (8) the formal concepts contained in $\mathcal{B}(X^*, Y^*, I)$ are in a correspondence with the classical formal concepts [18] of the crisp formal context $\langle X, Y, I^*\rangle$ (recall that crisp **L**-sets can be identitifed with the ordinary sets). Denoting the collection of classical formal concepts of $\langle X, Y, I^*\rangle$ by $\mathcal{B}(X, Y, I^*)$, we have

Theorem 4. *For a truth stresser * defined by (8), we have*

(i) *if $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$ then $\langle A^*, B^*\rangle \in \mathcal{B}(X, Y, I^*)$;*
(ii) *if $\langle C, D\rangle \in \mathcal{B}(X, Y, I^*)$ then $\langle C^{\uparrow*\downarrow}, D^{\downarrow*\uparrow}\rangle \in \mathcal{B}(X^*, Y^*, I)$.*

Proof. First, observe that if $A \in L^X$ is crisp then $A^{\uparrow*}(y) = 1$ iff for any $x \in X$: if $A(x) = 1$ then $I^*(x, y) = 1$. Analogously for crisp $B \in L^Y$. Hence, we can identify the ordinary Galois connection induced by $\langle X, Y, I^*\rangle$ with $\langle^{\uparrow*}, ^{\downarrow*}\rangle$.

(i) Let $A = B^{*\downarrow}$ and $B = A^{*\uparrow}$, i.e. $A^* = B^{*\downarrow*}$ and $B^* = A^{*\uparrow*}$ which give $\langle A^*, B^*\rangle \in \mathcal{B}(X, Y, I^*)$.

(ii) For crisp **L**-sets $C \in L^X$ and $D \in L^Y$ with $\langle C, D\rangle \in \mathcal{B}(X, Y, I^*)$ we have $C = D^{\downarrow*}$ and $D = C^{\uparrow*}$. Thus, $C^{\uparrow*\downarrow*\uparrow} = D^{\downarrow*\uparrow}$ and $D^{\downarrow*\uparrow*\downarrow} = C^{\uparrow*\downarrow}$, proving the assertion. □

Theorem 4 yields that the 1-cuts of concepts contained in $\mathcal{B}(X^*, Y^*, I)$ are exactly the classical concepts of $\langle X, Y, I^*\rangle$ (I^* itself is an 1-cut of the original $I \in L^X \times L^Y$). But unlike the concepts in $\mathcal{B}(X, Y, I^*)$, the concepts in $\mathcal{B}(X^*, Y^*, I)$ can contain an additional "fuzzy information" which is lost when considering the crisp context $\langle X, Y, I^*\rangle$.

Automatic generation of statements

The exploration of possible placements of truth stressers bring up the following aspect: we often construct proofs in which we use inequalities of the form $A^{\cdots} \subseteq A^{\cdots}$ (and dually for B), where "..." stand for sequences of $^{\uparrow}$, $^{\downarrow}$, and *. Such inequalities themselves are assertions that should be proven. The proofs of these inequalities are usually purely combinatorial and tedious. On the other hand a large "database" of such assertions can provide us with an essential insight into the modified Galois connections. For this purpose, we designed a computer program to find the proofs automatically.

For illustration, we present a segment of the assertions, found by the program, which is limited only to inequalities $B^{\cdots} \subseteq B^{\cdots}$ whose left and right side of "$\subseteq$" contain at most four symbols and we also skip trivial formulas like $B^{\downarrow} \subseteq B^{\downarrow}$, $B^{\downarrow\uparrow\downarrow} \subseteq B^{\downarrow}$, etc. Restricted by this limitation, the program generated 210 formulas from which 37 were proven (a table follows); for the remaining 173 ones the engine found a counterexample.

$B \subseteq B^{\downarrow*\uparrow}$	$B^{\downarrow} \subseteq B^{\downarrow\uparrow*\downarrow}$	$B^{*\downarrow\uparrow} \subseteq B^{\downarrow*\uparrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{*\downarrow*\uparrow}$
$B \subseteq B^{\downarrow\uparrow}$	$B^{\downarrow*} \subseteq B^{\downarrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{\downarrow\uparrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{\downarrow*\uparrow*}$
$B^{*} \subseteq B$	$B^{*\downarrow*} \subseteq B^{*\downarrow}$	$B^{\downarrow\uparrow*} \subseteq B^{\downarrow*\uparrow*}$	$B^{\downarrow*\uparrow*} \subseteq B^{\downarrow*\uparrow}$
$B^{*} \subseteq B^{\downarrow\uparrow}$	$B^{\downarrow*} \subseteq B^{*\downarrow*}$	$B^{\downarrow\uparrow*} \subseteq B^{\downarrow\uparrow}$	$B^{\downarrow*\uparrow\downarrow} \subseteq B^{*\downarrow}$
$B^{*} \subseteq B^{*\downarrow\uparrow}$	$B^{*} \subseteq B^{\downarrow*\uparrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{*\downarrow\uparrow}$	$B^{\downarrow*\uparrow\downarrow} \subseteq B^{\downarrow\uparrow*\downarrow}$
$B^{*} \subseteq B^{\downarrow\uparrow*}$	$B^{\downarrow*} \subseteq B^{*\downarrow}$	$B^{*\downarrow*\uparrow} \subseteq B^{\downarrow*\uparrow}$	$B^{\downarrow*\uparrow\downarrow} \subseteq B^{\downarrow}$
$B^{*} \subseteq B^{*\downarrow\uparrow*}$	$B^{\downarrow*} \subseteq B^{\downarrow*\uparrow\downarrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{\downarrow\uparrow*}$	$B^{\downarrow\uparrow*\downarrow} \subseteq B^{*\downarrow}$
$B^{\downarrow} \subseteq B^{*\downarrow}$	$B^{\downarrow*} \subseteq B^{\downarrow\uparrow*\downarrow}$	$B^{*\downarrow\uparrow} \subseteq B^{*\downarrow*\uparrow}$	
$B^{*} \subseteq B^{*\downarrow*\uparrow}$	$B^{*\downarrow\uparrow} \subseteq B^{\downarrow\uparrow}$	$B^{\downarrow\uparrow*} \subseteq B^{\downarrow*\uparrow}$	
$B^{*} \subseteq B^{\downarrow*\uparrow*}$	$B^{\downarrow\uparrow} \subseteq B^{\downarrow*\uparrow}$	$B^{*\downarrow\uparrow*} \subseteq B^{\downarrow*\uparrow}$	

A large database of 375 assertions (with the proofs attached) can be found at `http://vychodil.inf.upol.cz/res/devel/aureas`. The general inference engine is still under construction [17] and will be available soon at the same Internet address.

4 Epilogue: Applications and Further Development

We presented motivations and introductory results on fuzzy Galois connections with truth stressers. We showed that for Baaz truth stresser, Galois connections with truth stressers are exactly the "Galois-like-connections" behind the so-called crisply generated formal concepts obtained from object-attribute data with fuzzy attributes. Let us now demonstrate the reduction of the number of extracted concepts from the object-attribute data $\langle X, Y, I\rangle$—the main effect of considering $\mathcal{B}(X, Y^{*}, I)$ (crisply generated concepts) instead of $\mathcal{B}(X, Y, I)$ (all formal concepts).

The following experiment is taken from [14]. The experiment demonstrates the factor of reduction, i.e. the ratio $r = |\mathcal{B}(X, Y^{*}, I)|/|\mathcal{B}(X, Y, I)|$ (the smaller, the larger the reduction). Tab. 1 shows the values of r for 10 experiments (columns) run over randomly generated formal contexts $\langle X, Y, I\rangle$ (rows) with the number of objects equal to the number of attributes (from 5 to 25 objects/attributes) and with $|L| = 11$ (11 truth degrees). Moreover, we show average and dispersion of r. We can see that the dispersion is relatively low and that r decreases with growing size of data.

The future research is needed in the following directions:

	1	2	3	4	5	6	7	8	9	10	Av	Var
5	0,58	0,4	0,38	0,53	0,48	0,38	0,43	0,41	0,48	0,33	0,441	0,0733
6	0,31	0,31	0,38	0,43	0,38	0,32	0,43	0,42	0,36	0,38	0,372	0,0443
7	0,46	0,37	0,31	0,48	0,45	0,27	0,41	0,43	0,4	0,37	0,395	0,0635
8	0,35	0,26	0,32	0,33	0,29	0,3	0,27	0,34	0,3	0,31	0,308	0,0270
...	...	...	...	...	...	...	...	...	...	...	...	...
22	0,1	0,09	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,097	0,0038
23	0,09	0,11	0,1	0,1	0,1	0,1	0,1	0,09	0,1	0,11	0,099	0,0066
24	0,1	0,09	0,08	0,1	0,09	0,1	0,09	0,09	0,09	0,08	0,090	0,0079
25	0,08	0,07	0,09	0,08	0,09	0,09	0,08	0,07	0,09	0,08	0,081	0,0074

Table 1. Behavior of r (its average Av and dispersion Var) in dependence on size of input data table (rows; the numbers 5 . . . 25 denote the number of objects, this is equal to the number of attributes); columns correspond to experiments.

- more detailed investigation of fuzzy Galois connection with truth stressers with other placement of * than in (9) and (10);
- extension of the results to other truth stressers than (8);
- study of the systems of fixed points of fuzzy Galois connections with truth stressers (i.e. restricted concept lattices) with various placements of *, and their relationship (e.g. show the analogy/generalization of Wille's main theorem of concept lattices [29], see also [13] for fuzzy setting);
- algorithms for generating fixed points, i.e. formal concepts, of Galois connections with truth stressers.

As already demonstrated, better understanding of the presented issues can enhance the applicablity of formal concept analysis of data with fuzzy attributes.

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