Confluence of Fuzzy Relations over Similarity Spaces*

Tomas Kuhr
Dept. Computer Science
Palacky University, Olomouc
Olomouc, CZ–771 46
Czech Republic
tomas.kuhr@upol.cz

Vilem Vychodil†
Dept. Computer Science
Palacky University, Olomouc
Olomouc, CZ–771 46
Czech Republic
vilem.vychodil@upol.cz

Abstract

This paper presents a preliminary study of confluence and related properties of fuzzy relations on similarity spaces. Confluence is an essential property of relations connected to the idea of rewriting and substituting which appear in abstract rewriting systems. This paper is a continuation of our previous results showing that confluence, termination, and related properties can be introduced as properties of fuzzy relations using residuated structures of truth degrees, leaving the ordinary notions a particular case when the underlying structure of truth degrees is the two-valued Boolean algebra. In this paper, we focus on analogous properties of fuzzy relations defined on sets equipped with similarity fuzzy relations.

1 Introduction and Problem Setting

In the theory of abstract rewriting systems, the confluence and termination are properties of binary relations which are used to perform substitutions specified by the respective binary relation. Since their inception, the notions have been the subject of extensive investigations and many applications of abstract rewriting systems evolved. For instance, term rewriting systems can be used as a theoretical background for functional programming, various logical deductive systems can be formalized by rewriting systems, rewriting plays an important role in the theory of formal grammars, etc. Good overview of rewriting systems can be found in [Baader, Nipkow, 1999] and [Wechler, 1992].

Confluence and Termination Confluence and termination can be introduced as follows. Let R be a binary relation on a set X and assume that $\langle x, y\rangle \in R$ means that one may substitute y for x. The substitution "y for x" can be informally explained so that, from a certain point of view, whenever x does a certain job, y does it as well. An element $x \in X$ is called reducible if $\langle x, y\rangle \in R$ for some $y \in Y$; otherwise, x is called irreducible. By a reduction we mean any sequence $x_0, \ldots, x_n$ such that $\langle x_{i-1}, x_i\rangle \in R$ ($i = 1, \ldots, n$); a reduction is called terminating if x_n is irreducible. In this case, x_0 is said to be reducible to x_n. Relation R is called *terminating* if it has only terminating reductions. Relation R is called *confluent* whenever x is reducible to both y and y' then there is some z such that both y and y' are reducible to z. Termination and confluence obey several interesting properties. There is a synergy between termination and confluence the most known example of which is that a relation which is both terminating and confluent has normal forms, i.e. each element is reducible to a unique irreducible element.

Graded Substitutability The main motivation of this study is the fact that there are natural examples where the notion of substitutability is inherently fuzzy rather than crisp. Therefore, we look at substitutability and the properties of confluence and termination from the point of view of fuzzy logic [Belohlavek, 2002; Gottwald, 2001; Hájek, 1998] and fuzzy set theory [Zadeh, 1965]. Our basic motivation is the fact that the phenomenon of substitutability may not be bivalent. It is a common practice of everyday life to substitute y for x whenever x is too complex to handle and y does the job of x sufficiently well. In a similar way, one often substitutes an option y for option x whenever y is much cheaper than x and y does the job of x sufficiently well. For example, instead of using an expensive option poll based on survey of a sample of $10,000$ persons, one may use cheaper option poll based on few hundreds of persons (which will give sufficiently similar result); instead of buying a 4 BR family house one may wish to buy a 3 BR house which is more affordable, etc. At the conceptual level, the common point of these examples is that one works with a substitutability relation R which is, however, a fuzzy relation rather than a bivalent one. That is, for every $x, y \in X$, $R(x, y)$ is the degree to which y can be substituted for x (not necessarily 0 or 1). $R(x, y)$ embodies the expert knowledge and depends on the particular situation.

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†The author is also with T. J. Watson School, SUNY Binghamton, Vestal, NY 13902–6000, USA.

Substitutability and Similarity Spaces The paper develops ideas which have been announced in [Belohlavek, Kuhr, Vychodil, 2009a] and are further studied in [Belohlavek, Kuhr, Vychodil, 2009b]. The main motivation for the present paper is the fact that, in many cases, the universe of discourse is equipped with a similarity relation (i.e., a fuzzy relation which is reflexive, symmetric, and transitive) prescribing degrees to which elements are similar. Therefore, the definition of a degree to which "y is substitutable for x" that has been introduced in the earlier papers can be extended so that the similarity is taken into account. Namely, we can define a degree to which "there are z_1 and z_2 such that x is similar to z_1 and z_2 is substitutable for z_1 and z_2 is similar to y" as a more general degree of substitutability with respect to the underlying similarity. Clearly, such an approach can yield more natural results then the original approach especially in cases where there is no y substitutable for x but there are some z_1 and z_2 which are (very) similar to x and y respectively such that z_2 is substitutable for z_1 to a high degree.

Therefore, we will introduce notions related to substitutability given by a fuzzy relations defined on a similarity space (a set equipped with a similarity fuzzy relation, see [Belohlavek, 2002]) instead of just a set of elements. From the theoretical point of view, dealing with substitutability issues on similarity spaces should be interesting from several points of view. In addition to the above-mentioned motivation, the issue should also be relevant in the context of general algebra and term rewriting. For instance, in [Belohlavek, Vychodil, 2005; 2006a; 2006b] it is shown that the fundamental notions and results on the equational reasoning can be accommodated to the requirement of respecting in a natural way an underlying similarity relation on the universe set. Given an underlying similarity, one may require (or find out) two terms to be substitutable only to some degree which may result from the fact that they always evaluate to elements which are similar but not identical. Hence, the investigation of graded substitutability on similarity spaces can bring new insight into the graded equational reasoning.

Because of the limited scope of this paper, we do not include full proofs for all observations we present. Details are postponed to a full version of this paper.

2 Preliminaries

Structures of Truth Degrees We use complete residuated lattices as structures of truth values. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and $a \otimes 1 = a$); (ii) $\otimes$ (conjunction) and $\rightarrow$ (residuum) are binary operations satisfying the adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is true for all $a, b, c \in L$. In what follows, $\mathbf{L}$ always refers to a complete residuated lattice. Note that the class of complete residuated lattices includes structures of truth degrees on the real unit interval with $\otimes$ and $\rightarrow$ being a left-continuous t-norm and its corresponding residuum, respectively. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:}\quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \qquad (1)$$

$$\text{Gödel:}\quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise,} \end{cases} \end{aligned} \qquad (2)$$

$$\text{Goguen:}\quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{otherwise.} \end{cases} \end{aligned} \qquad (3)$$

Recall that the two-valued Boolean algebra which plays an important role in the classical logic is a particular case of a complete residuated lattice where $L = \{0, 1\}$, $\wedge$ and $\vee$ being the minimum and maximum, respectively, $\otimes = \wedge$, and $\rightarrow$ being truth function of the two-valued implication. In the sequel, the two-valued Boolean algebra will be denoted by $\mathbf{2}$. When necessary, we shall consider additional properties of truth degrees. Namely, $a \in L$ is called idempotent iff $a \otimes a = a$. Furthermore, $0 \neq a \in L$ is called a zero-divisor of $\otimes$ if there is $0 \neq b \in L$ such that $a \otimes b = 0$. Thus, $\mathbf{L}$ has no zero-divisors iff for any $a, b \in L$: if $a \otimes b = 0$ then $a = 0$ or $b = 0$. Directly from (2), if $\otimes$ is a Gödel conjunction then each $a \in L$ is idempotent; if $\otimes$ is the Łukasiewicz conjunction on $L = [0, 1]$ then each $0 < a < 1$ is a zero divisor, see (1). All properties of complete residuated lattices used in the sequel are well known and can be found in [Belohlavek, 2002; Gottwald, 2001; Hájek, 1998].

Fuzzy Sets and Relations We now briefly recall basic notions of fuzzy sets and fuzzy relations. Let $\mathbf{L}$ be a complete residuated lattice. An $\mathbf{L}$-set (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set X is any map $A\colon X \rightarrow L$, $A(x) \in L$ being interpreted as the truth value of "x belongs to A". Analogously, an n-ary $\mathbf{L}$-relation (or fuzzy relation with truth degrees in $\mathbf{L}$) on a universe set X is an $\mathbf{L}$-set in the universe set X^n, e.g. a binary $\mathbf{L}$-relation R on X is a map $R\colon X \times X \rightarrow L$. Binary $\mathbf{L}$-relations will be denoted by capital letters $R, R', \ldots$ or symbols $\rightharpoonup, \leftharpoondown, \ldots$ in which case we write $u \rightharpoonup v$ and $u \leftharpoondown v, \ldots$ instead of $\rightharpoonup(u, v)$ and $\leftharpoondown(u, v) \ldots$, respectively.

For binary $\mathbf{L}$-relations R_1, R_2 on X the $\circ$-composition of R_1 and R_2 is a binary $\mathbf{L}$-relation $R_1 \circ R_2$ on X defined by

$$(R_1 \circ R_2)(x, y) = \textstyle\bigvee_{z \in X} \big(R_1(x, z) \otimes R_2(z, y)\big). \qquad (4)$$

For $\mathbf{L}$-sets A and B in X we define degrees $S(A, B) \in L$ and $E(A, B) \in L$ as follows:

$$S(A, B) = \textstyle\bigwedge_{x \in X} (A(x) \rightarrow B(x)), \qquad (5)$$

$$E(A, B) = \textstyle\bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \qquad (6)$$

$S(A, B)$ is called a degree of subsethood of A in B; $E(A, B)$ is called a degree of equality of A and B.

Note, that $a \leftrightarrow b$ in (6) is an abbreviation for $(a \rightarrow b) \wedge (b \rightarrow a)$. Clearly, $E(A, B) = S(A, B) \wedge S(B, A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A, B) = 1$, i.e. if $A(x) \leq B(x)$ is true for each $x \in X$.

For any **L**-set $A\colon X \to L$ and $a \in L$, we define its strong a-cut ${}^{a+}/A \subseteq X$ by ${}^{a+}/A = \{x \in X \mid A(x) > a\}$. In particular, the strong 0-cut ${}^{0+}/A$ of A is ${}^{0+}/A = \{x \in X \mid A(x) > 0\}$.

Similarity Spaces A binary **L**-relation $\approx\colon X \times X \to L$ is called an **L**-similarity (shortly, a similarity) if it is (i) reflexive, i.e., $x \approx x = 1$ for each $x \in X$, (ii) symmetric, i.e., $x \approx y = y \approx x$ for each $x, y \in X$, and (iii) $\otimes$-transitive, i.e., $x \approx y \otimes y \approx z \leq x \approx z$ for each $x, y, z \in X$. A set X with an **L**-similarity $\approx\colon X \times X \to L$, denoted $\langle X, \approx \rangle$, is called a similarity space, see [Belohlavek, 2002]. A binary **L**-relation R on X is called compatible with $\approx$ if, for each $x_1, x_2, y_1, y_2 \in X$:

$$x_1 \approx x_2 \otimes y_1 \approx y_2 \otimes R(x_1, y_1) \leq R(x_2, y_2).$$

In the sequel, we consider a similarity space $\langle X, \approx \rangle$ and for fuzzy relations defined on X we introduce notions related to substitutability and investigate their properties. Interestingly, we will take a *reductionist approach* and show that the newly introduced notions can be reduced to the notions introduced in [Belohlavek, Kuhr, Vychodil, 2009a; 2009b].

3 Reductions

In this section, we define graded counterpart of the bivalent notion of a reduction and investigate its properties.

In what follows, $\langle X, \approx \rangle$ is an **L**-similarity space and $\rightharpoonup$ a binary **L**-relation on X. Given $\rightharpoonup$ and $\approx$, we define a degree to which a element from X can be reduced to another element from X with respect to $\rightharpoonup$ and $\approx$:

Definition 1. For $x, y \in X$ we define a degree $x \rightharpoonup^*_\approx y \in L$ to which the element x can be reduced to the element y by

$$x \rightharpoonup^*_\approx y = \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_{2k} \rangle \in X^{2N}} \big(x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k} \approx y \big),$$

where $X^{2N} = \bigcup_{n \in N_0} X^{2n}$, i.e. X^{2N} is a union of all even Cartesian powers of the set X.

According to Definition 1 the degree $x \rightharpoonup^*_\approx y$ can be seen as a degree to which "there are some elements $z_1, z_2, \ldots, z_{2k}$ in X such that x is equal to z_1 and z_1 reduces to z_2 and z_2 is equal to z_3 and, ..., and z_{2k-1} reduces to z_{2k} and z_{2k} is equal to y".

Definition 1 generalizes the notion of reduction $\rightharpoonup^*$ that has been introduced in [Belohlavek, Kuhr, Vychodil, 2009a]. Namely, if one considers a trivial **L**-similarity space $\langle X, \approx \rangle$ where $\approx$ is the crisp equality (i.e., the identity) then obviously $x \rightharpoonup^*_\approx y = x \rightharpoonup^* y$, where $\rightharpoonup^*$ is a reflexive and transitive closure of $\rightharpoonup^*$.

Moreover, if $\mathbf{L} = \mathbf{2}$ and if $\approx$ is the crisp equality relation then $x \rightharpoonup^*_\approx y = 1$ iff $x_0, \ldots, x_n$ is a reduction in the usual sense, see [Baader, Nipkow, 1999].

Following the discussion in Section 1, we can easily see that $x \approx y \leq x \rightharpoonup^*_\approx y$ for all elements $x, y \in X$. Thus, $\approx \subseteq \rightharpoonup^*_\approx$. This is a consequence of $X^0 \subseteq X^{2N}$.

We can prove that in case of relations $\rightharpoonup$ compatible with $\approx$, the corresponding $\rightharpoonup^*_\approx$ simplifies:

Theorem 2. *If $\rightharpoonup$ is compatible with $\approx$ then $x \rightharpoonup^*_\approx y = x \rightharpoonup^* y$, where $\rightharpoonup^*$ denotes the reflexive and transitive closure of $\rightharpoonup$.*

Proof. Since $\approx$ is reflexive and $\otimes$-transitive, we get

$$u \approx w = \bigvee\nolimits_v (u \approx v \otimes v \approx w).$$

Thus, we may write

$$x \rightharpoonup^*_\approx y = \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_{2k} \rangle} \big(x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3 \otimes z_3 \rightharpoonup z_4 \otimes z_4 \approx z_5 \otimes \cdots \otimes z_{2k} \approx y \big) =$$

$$= \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_{3k-1} \rangle} \big((x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3) \otimes (z_3 \approx z_4 \otimes z_4 \rightharpoonup z_5 \otimes z_5 \approx z_6) \otimes \cdots \otimes (z_{3k-3} \approx z_{3k-2} \otimes z_{3k-2} \rightharpoonup z_{3k-1} \otimes z_{3k-1} \approx y \big).$$

Using the compatibility of a binary **L**-relation $\rightharpoonup$ and an **L**-similarity $\approx$, the inequality

$$u_1 \approx v_1 \otimes u_2 \approx v_2 \otimes v_1 \rightharpoonup v_2 \leq u_1 \rightharpoonup u_2$$

holds for every $u_1, u_2, v_1, v_2 \in X$. As a consequence,

$$u_1 \rightharpoonup u_2 = \bigvee\nolimits_{v_1, v_2} (u_1 \approx v_1 \otimes u_2 \approx v_2 \otimes v_1 \rightharpoonup v_2).$$

Therefore, $x \rightharpoonup^*_\approx y$ can be simplified as follows

$$x \rightharpoonup^*_\approx y =$$

$$= \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_{3k-1} \rangle} (x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3) \otimes \cdots$$

$$\cdots \otimes (z_{3k-3} \approx z_{3k-2} \otimes z_{3k-2} \rightharpoonup z_{3k-1} \otimes z_{3k-1} \approx y) =$$

$$= \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_k \rangle} x \rightharpoonup z_1 \otimes z_1 \rightharpoonup z_2 \otimes \cdots z_k \rightharpoonup y =$$

$$= x \rightharpoonup^* y,$$

which proves the claim. □

As a consequence, if $\rightharpoonup$ is compatible with $\approx$ then $\rightharpoonup^*$ is identical to $\rightharpoonup^*_\approx$. A question remains if, in general, for each $\rightharpoonup$ one can find $\rightharpoonup_\circ$ such that $\rightharpoonup^*_\approx$ equals $(\rightharpoonup_\circ)^*$ where $(\rightharpoonup_\circ)^*$ is a reflexive and transitive closure of $\rightharpoonup$. The following lemma will show that it is indeed the case:

Lemma 3. *For $\rightharpoonup_\circ$ defined by $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$, we have $u \rightharpoonup^*_\approx v = u \rightharpoonup_\circ{}^* v$ for all $u, v \in X$.*

Proof. Inspect the proof of Theorem 2 and observe that $x \rightharpoonup^*_{\approx} y$ can be written as

$$\bigvee_{\langle z_1,z_2,\ldots,z_{3k-1}\rangle}\big((x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3) \otimes (z_3 \approx z_4 \otimes z_4 \rightharpoonup z_5 \otimes z_5 \approx z_6) \otimes \cdots \otimes (z_{3k-3} \approx z_{3k-2} \otimes z_{3k-2} \rightharpoonup z_{3k-1} \otimes z_{3k-1} \approx y)\big) =$$

$$\bigvee_{\langle z_3,z_6,\ldots,z_{3k-3}\rangle}\big(\bigvee_{z_1,z_2}(x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3) \otimes \bigvee_{z_4,z_5}(z_3 \approx z_4 \otimes z_4 \rightharpoonup z_5 \otimes z_5 \approx z_6) \otimes \cdots \otimes \bigvee_{z_{3k-2},z_{3k-1}}(z_{3k-3} \approx z_{3k-2} \otimes z_{3k-2} \rightharpoonup z_{3k-1} \otimes z_{3k-1} \approx y)\big).$$

The rest is obvious. □

4 Confluence of Fuzzy Relations

In this section, we introduce confluence of **L**-relations based on the notions of divergence and convergence. Recall that in the classical case, elements x and y are convergent if they are reducible (according to a binary relation $\rightsquigarrow$) to a common element, whereas elements x and y are divergent if there is an element which is reducible to both x and y. In our setting, we define the notions as follows:

Definition 4. For $x, y \in X$ we define $x \downharpoonleft_{\approx} y$ by

$$x \downharpoonleft_{\approx} y = \bigvee_{z_1,z_2 \in X}(x \rightharpoonup^*_{\approx} z_1 \otimes y \rightharpoonup^*_{\approx} z_2 \otimes z_1 \approx z_2).$$

$x \downharpoonleft_{\approx} y$ is called the ***degree of convergence*** of x and y with respect to $\rightharpoonup$ and $\approx$. We also say that x and y are convergent (with respect to $\rightharpoonup$ and $\approx$) to degree $x \downharpoonleft_{\approx} y$. If $x \downharpoonleft_{\approx} y = 1$ we say that x and y are convergent.

In case of a fuzzy relation compatible with $\approx$, we can provide a similar characterization of convergence degrees as in Theorem 2. Namely,

Theorem 5. *If $\rightharpoonup$ is compatible with $\approx$ then*

$$x \downharpoonleft_{\approx} y = \bigvee_{z \in X}(x \rightharpoonup^* z \otimes y \rightharpoonup^* z),$$

i.e. $\downharpoonleft_{\approx}$ coincides with $\downharpoonleft$ defined in [Belohlavek, Kuhr, Vychodil, 2009a].

Proof. Uses similar arguments as in Theorem 2. □

If $\rightharpoonup$ is not compatible with $\approx$, we can define an **L**-relation $\rightharpoonup_\circ$ as in Lemma 3 whose degree of convergence $x \downharpoonleft_\circ y$ (without considering any **L**-similarity, i.e., as defined in [Belohlavek, Kuhr, Vychodil, 2009a]) and $x \downharpoonleft_{\approx} y$ (degree of convergence with respect to $\rightharpoonup$ and $\approx$) are equal:

Theorem 6. *For $\rightharpoonup_\circ$ defined by $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$, we have $u \downharpoonleft_{\approx} v = u \downharpoonleft_\circ v$ for all $u, v \in X$.*

Proof. Follows from Theorem 5 and Lemma 3. □

As in case of convergence, one can define graded notions of divergence of fuzzy relations over similarity spaces:

Definition 7. For $x, y \in X$ we define $x \upharpoonleft_{\approx} y$ by

$$x \upharpoonleft_{\approx} y = \bigvee_{z_1,z_2 \in X}(z_1 \rightharpoonup^*_{\approx} x \otimes z_2 \rightharpoonup^*_{\approx} y \otimes z_1 \approx z_2).$$

$x \upharpoonleft_{\approx} y$ is called the ***degree of divergence*** of x and y with respect to $\rightharpoonup$ and $\approx$. We also say that x and y are divergent (with respect to $\rightharpoonup$ and $\approx$) to degree $x \upharpoonleft_{\approx} y$ or just divergent if $x \upharpoonleft_{\approx} y = 1$.

Analogous observations to those from Theorem 5 and Theorem 6 can be made:

Theorem 8. *If $\rightharpoonup$ is compatible with $\approx$ then*

$$x \upharpoonleft_{\approx} y = \bigvee_{z \in X}(z \rightharpoonup^* x \otimes z \rightharpoonup^* y),$$

i.e., $\upharpoonleft_{\approx}$ coincides with $\upharpoonleft$ defined in [Belohlavek, Kuhr, Vychodil, 2009a]. Moreover, for any $\rightharpoonup$, $\approx$, and the corresponding $\rightharpoonup_\circ$, we have $u \downharpoonleft_{\approx} v = u \downharpoonleft_\circ v$ for all $u, v \in X$.

Proof. Dual to Theorem 5 and Theorem 6. □

Confluence is defined using convergence and divergence in much the same way as in the ordinary case:

Definition 9. The degree $\mathrm{CFL}(\rightharpoonup)_{\approx}$ to which $\rightharpoonup$ is ***confluent*** with respect to $\rightharpoonup$ and $\approx$ is defined by $\mathrm{CFL}(\rightharpoonup)_{\approx} = S(\upharpoonleft_{\approx}, \downharpoonleft_{\approx})$. If $\mathrm{CFL}(\rightharpoonup)_{\approx} = 1$ we say that $\rightharpoonup$ is confluent.

Described verbally, the degree of confluence of $\rightharpoonup$ w.r.t. $\approx$ is a degree to which the following is true: "if any x and y are divergent then x and y are convergent". As one may expect, degrees of confluence can be expressed as degrees of confluence of the corresponding $\rightharpoonup_\circ$:

Theorem 10. *For any $\rightharpoonup$ and $\approx$, we have*

$$\mathrm{CFL}(\rightharpoonup)_{\approx} = \mathrm{CFL}(\rightharpoonup_\circ)$$

where $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$. Moreover, if $\rightharpoonup$ is compatible with $\approx$ then $\mathrm{CFL}(\rightharpoonup)_{\approx} = \mathrm{CFL}(\rightharpoonup) = S(\upharpoonleft, \downharpoonleft)$.

Proof. Follows from [Belohlavek, Kuhr, Vychodil, 2009a] and previous assertions. □

5 Properties of Confluence

In this section, we shall investigate properties of confluence. The main result shows that one can define a degree to which $\rightharpoonup$ has a Church-Rosser property and we put in correspondence degrees of confluence and degrees to which $\rightharpoonup$ has the Church-Rosser property.

In order to define the graded Church-Rosser property, we consider convertibility degrees of fuzzy relations over similarity spaces: for $x, y \in X$, we define the degree $x \rightleftharpoons^*_{\approx} y$ to which x and y are *convertible* with respect to $\rightharpoonup$ and $\approx$ by

$$x \rightleftharpoons^*_{\approx} y = \bigvee_{\langle z_1,z_2,\ldots,z_{2k}\rangle \in X^{2N}}\big(x \approx z_1 \otimes z_1 \rightleftharpoons z_2 \otimes \otimes z_2 \approx z_3 \otimes \cdots z_{2k-1} \rightleftharpoons z_{2k} \otimes z_{2k} \approx y\big),$$

where (i) $X^{2N} = \bigcup_{n \in N_0} X^{2n}$, i.e. X^{2N} is a union of all even Cartesian powers of the set X and (ii) $\rightleftharpoons = \rightharpoonup \cup \rightharpoonup^{-1}$, i.e. $\rightleftharpoons$ is the symmetric closure of $\rightharpoonup$.

Again, similar observations as before can be made about the convertibility degrees $\rightleftharpoons^*_\approx$ and $\rightleftharpoons^*_\circ$. We now introduce the following notion:

Definition 11. The degree $\mathrm{CR}(\rightharpoonup)_\approx$ to which $\rightharpoonup$ has the ***Church-Rosser property*** w.r.t. $\rightharpoonup$ and $\approx$ is defined by $\mathrm{CR}(\rightharpoonup)_\approx = S(\rightleftharpoons^*_\approx, \downharpoonleft_\approx)$. If $\mathrm{CR}(\rightharpoonup)_\approx = 1$ we say that $\rightharpoonup$ has the Church-Rosser property.

Theorem 12. *For any $\rightharpoonup$ and $\approx$, we have*

$$\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ)$$

where $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$. Moreover, if $\rightharpoonup$ is compatible with $\approx$ then $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup) = S(\rightleftharpoons^*, \downharpoonleft)$.

Proof. Using previous observations. □

We can prove the following characterizations:

Theorem 13. $\mathrm{CR}(\rightharpoonup)_\approx = E(\rightleftharpoons^*_\approx, \downharpoonleft_\approx)$.

Proof. Take $\rightharpoonup_\circ$ and apply previous observations together with a corresponding claim from [Belohlavek, Kuhr, Vychodil, 2009a]. Then,

$$\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ) = E(\rightleftharpoons^*_\circ, \downharpoonleft_\circ) = E(\rightleftharpoons^*_\approx, \downharpoonleft_\approx),$$

which proves the claim. □

The relationship between degrees of confluence and degrees to which fuzzy relations have the Church-Rosser property is characterized by the following:

Theorem 14. *If $\mathrm{CFL}(\rightharpoonup)_\approx$ is an idempotent element of $\mathbf{L}$ then $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup)_\approx$.*

Proof. Consider $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$. Using previous observations, we get $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ)$ and $\mathrm{CFL}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup_\circ)$ hold. The rest follows by applying the corresponding assertion from [Belohlavek, Kuhr, Vychodil, 2009a] to $\mathrm{CR}(\rightharpoonup_\circ)$ and $\mathrm{CFL}(\rightharpoonup_\circ)$. Thus,

$$\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ) = \mathrm{CFL}(\rightharpoonup_\circ) = \mathrm{CFL}(\rightharpoonup)_\approx,$$

proving the claim. □

If $\otimes = \wedge$ then $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup)_\approx$ which is an obvious corollary of Theorem 14. Moreover, one can claim that an $\mathbf{L}$-relation $\rightharpoonup$ has the Church-Rosser property with respect to an $\mathbf{L}$-similarity $\approx$ iff it is confluent with respect to $\approx$. This follows immediately from the fact that 1 is idempotent.

We have shown in this section that confluence of fuzzy relations over similarity spaces possesses analogous properties as the ordinary confluence of binary relations. Also note that the results were obtained using a reductionist approach using the fact that instead of considering $\rightharpoonup$ over $\langle X, \approx \rangle$ one may take $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$ over X. Another observation that we have made is that taking an $\rightharpoonup$ which is compatible with $\approx$ yields exactly the same results if we take $\rightharpoonup$ and completely neglect $\approx$.

6 Termination of Fuzzy Relations

The aim of this section is to present notes on termination of fuzzy relations over similarity spaces. Termination is considered an important property of abstract rewriting systems. Intuitively, termination can be seen as a natural property of "algorithms", saying that each reduction may terminate after finitely many steps. In case of fuzzy relations defined over similarity spaces, the notion of a termination becomes more complex.

We start by introducing the following notation. For $x_0, \ldots, x_n \in X$ we define a degree $\mathrm{re}(x_0, \ldots, x_n)_\approx \in L$ by induction as follows

(i) $\mathrm{re}(x_0, \ldots, x_n)_\approx = 1$ if $n = 0$,

(ii) $\mathrm{re}(x_0, \ldots, x_n)_\approx = \mathrm{re}(x_0, \ldots, x_{n-1})_\approx \otimes x_{n-1} \approx x_n$ if n is odd,

(iii) $\mathrm{re}(x_0, \ldots, x_n)_\approx = \mathrm{re}(x_0, \ldots, x_{n-1})_\approx \otimes x_{n-1} \rightharpoonup x_n$ otherwise.

$\mathrm{re}(x_0, \ldots, x_n)_\approx$ is called a degree to which $x_0, \ldots, x_n$ is a reduction with respect to $\rightharpoonup$ and $\approx$.

Remark 15. Observe that $\mathrm{re}(x_0)_\approx = 1$, $\mathrm{re}(x_0, x_1)_\approx = x_0 \approx x_1$, $\mathrm{re}(x_0, x_1, x_2)_\approx = x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2$, $\mathrm{re}(x_0, x_1, x_2, x_3)_\approx = x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2 \otimes x_2 \approx x_3$, and $\mathrm{re}(x_0, \ldots, x_4)_\approx = x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2 \otimes x_2 \approx x_3 \otimes x_3 \rightharpoonup x_4$, etc. Hence, $\mathrm{re}(x_0, \ldots, x_n)_\approx$ is a degree to which x_0 is similar to x_1 and x_1 can be substituted by x_2 and x_2 is similar to x_3 which can be substituted by $x_4, \ldots$ Clearly, if $\mathbf{L}$ is the two-value Boolean algebra and if $\approx$ is the identity relation then $\mathrm{re}(x_0, \ldots, x_n)_\approx = 1$ means that x_0 is reducible to x_n in the classical sense.

For a sequence $x_0, \ldots, x_n$ of elements from X, we denote $\mathrm{nt}(x_0, \ldots, x_n)_\approx \in L$ a degree defined by

(i) $\bigvee_{y \in X} (\mathrm{re}(x_0, \ldots, x_n)_\approx \otimes x_n \rightharpoonup y)$ if n is odd,

(ii) $\bigvee_{y_0, y_1 \in X} (\mathrm{re}(x_0, \ldots, x_n)_\approx \otimes x_n \approx y_0 \otimes y_0 \rightharpoonup y_1)$ otherwise.

Remark 16. Analogous observations as in Remark 15 can be made for $\mathrm{nt}(x_0, \ldots, x_n)_\approx$. In words, $\mathrm{nt}(x_0, \ldots, x_n)_\approx$ is a degree to which x_0 reduces to x_n and x_n is further reducible to another element.

Based on the previous notions, we can introduce terminating and strictly terminating reductions.

Definition 17. An element $x \in X$ has a ***terminating reduction*** with respect to $\rightharpoonup$ and $\approx$ if there is a finite sequence $x = x_0, x_1, \ldots, x_n$ ($n \geq 0$) such that $\mathrm{re}(x_0, \ldots, x_n)_\approx \neq 0$, and $\mathrm{nt}(x_0, \ldots, x_n)_\approx = 0$. An element $x \in X$ is called ***irreducible*** (w.r.t. $\rightharpoonup$ and $\approx$) if $\mathrm{nt}(x)_\approx = 0$. An element $x \in X$ has a ***strictly terminating reduction*** (w.r.t. $\rightharpoonup$ and $\approx$) if there is a terminating reduction $x = x_0, x_1, \ldots, x_n$, where x_n is irreducible (w.r.t. $\rightharpoonup$ and $\approx$). An element $x \in X$ has a ***nonterminating reduction*** if there is an infinite sequence $x = x_0, x_1, \ldots$ such that for each $n \in \mathbb{N}_0$, $\mathrm{nt}(x_0, \ldots, x_n)_\approx \neq 0$.

Since termination and strict termination of fuzzy relations have been introduced as bivalent notions, we should investigate their relationship to the (ordinary) termination of bivalent relations. The following assertion shows that termination of a fuzzy relation $\rightharpoonup$ can be observed from the strong 0-cut of $\rightharpoonup_\circ$ (defined by $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$). In the following, let $\rightsquigarrow_\circ$ denote the strong 0-cut of $\rightharpoonup_\circ$.

Theorem 18. *The following are true for any* **L**-*relation* $\rightharpoonup$ *and any* **L**-*similarity* $\approx$ *on* X.

(*i*) *If an element* x *has a strictly terminating reduction, then* x *has a terminating reduction in the strong* 0-*cut* $\rightsquigarrow_\circ$ *of* $\rightharpoonup_\circ$.

(*ii*) *If an element* x *has a nonterminating reduction, then* x *has a nonterminating reduction in the strong* 0-*cut* $\rightsquigarrow_\circ$ *of* $\rightharpoonup_\circ$.

Proof. (*i*): Let x have a strictly terminating reduction (w.r.t. $\rightharpoonup$ and $\approx$) $x = x_0, x_1, \ldots, x_n$. By definition, we have $\mathrm{re}(x_0, \ldots, x_n)_\approx \neq 0$ and $\mathrm{nt}(x_n)_\approx = 0$. The inequality $\mathrm{re}(x_0, \ldots, x_n)_\approx \neq 0$ yields $(x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2 \otimes x_2 \approx x_3) \otimes (x_3 \rightharpoonup x_4 \otimes x_4 \approx x_5) \otimes \cdots \otimes (x_{n-2} \rightharpoonup x_{n-1} \otimes x_{n-1} \approx x_n) \neq 0$ (for odd n) or $(x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2) \otimes (x_2 \approx x_3 \otimes x_3 \approx x_4) \otimes \cdots \otimes (x_{n-2} \approx x_{n-1} \otimes x_{n-1} \rightharpoonup x_n) \neq 0$ (for even n).

If n is odd, we have $(x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2 \otimes x_2 \approx x_3) \neq 0$ and $(x_i \rightharpoonup x_{i+1} \otimes x_{i+1} \approx x_{i+2}) \neq 0$ for each $i \in \{3, 5, \ldots, n-2\}$, i.e. $(x_0 \approx \circ \rightharpoonup \circ \approx x_3) \neq 0$ and $(x_i \rightharpoonup \circ \approx x_{i+2}) \neq 0$ for each $i \in \{3, 5, \ldots, n-2\}$. Using the fact $\rightharpoonup \circ \approx \subseteq \approx \circ \rightharpoonup \circ \approx$, we acquire that $x_0, x_3, x_5, \ldots, x_n$ is a terminating reduction in $\rightsquigarrow_\circ$. Analogous observation can be made if n is even.

(*ii*): Let x has a nonterminating reduction $x = x_0, x_1, \ldots$ By definition, $\mathrm{re}(x_0, \ldots, x_n)_\approx \neq 0$ holds for each $n \in \mathbb{N}_0$. Hence, $(x_0 \approx x_1 \otimes x_1 \rightharpoonup x_2) \otimes (x_2 \approx x_3 \otimes x_3 \rightharpoonup x_4) \otimes \cdots \neq 0$, i.e. $(x_i \approx x_{i+1} \otimes x_{i+1} \rightharpoonup x_{i+2}) \neq 0$ is true for each $i \in \{0, 2, 4, \ldots\}$. Therefore, for each $i \in \{0, 2, 4, \ldots\}$, we get $x_i \rightsquigarrow_\circ x_{i+2}$, i.e. $x_0, x_2, x_4 \ldots$ is a nonterminating reduction in the strong 0-cut $\rightsquigarrow_\circ$. □

We now introduce (strict) termination as a property of **L**-relations:

Definition 19. A binary **L**-relation $\rightharpoonup$ on X is called ***terminating*** with respect to $\approx$ if no $x \in X$ has a nonterminating reduction (w.r.t. $\approx$). Moreover, $\rightharpoonup$ is called ***strictly terminating*** with respect to $\approx$ if (i) $\rightharpoonup$ is terminating (w.r.t. $\approx$), and (ii) for each $x \in X$: if x has a terminating reduction $x = x_0, \ldots, x_n$ (w.r.t. $\approx$) then x_n is irreducible (w.r.t. $\approx$).

Theorem 20. *If* $\rightharpoonup$ *is terminating (w.r.t.* $\approx$*) then each* $x \in X$ *has a terminating reduction (w.r.t.* $\approx$*).*

Proof. Can be proved in a similar way as in [Belohlavek, Kuhr, Vychodil, 2009b]. □

One can introduce notions of $\rightharpoonup_\approx$-minimality and well-foundedness as of fuzzy relations with respect to $\approx$ and $\rightharpoonup$ in much the same way as in [Belohlavek, Kuhr, Vychodil, 2009b] and similar conclusions can be obtained. The results are postponed to a full version of this paper.

7 Conclusions

Presented was a preliminary study of confluence, termination, and related notions of fuzzy relations defined on similarity spaces. Future investigation will include relationship to well-foundedness and Noetherian induction. We also focus on further similarity issues and normal forms. An interesting problem is to find a closer connection to graded equational reasoning, see [Belohlavek, Vychodil, 2005; 2006a; 2006b].

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